The added-vector code of the odd-sign construction: maximality in dimensions 20 and 21, and bounds in dimension 19

Vince Gonzalez

Independent Researcher

ORCID: 0009-0005-3640-014X

United States

VinceGonzalez@me.com

Version 2

Abstract

Cohn and Li improved the known lower bounds for the kissing number in dimensions 17 through 21 by an odd-sign construction whose final ingredient is a binary code, here called the added-vector code, chosen inside a punctured extended binary Golay code. They remark that the limits of such constructions are unclear, and Ho subsequently improved dimension 19 by enlarging that code from 1024 to 1280 words. We show that the maximum size of the added-vector code is the independence number of an explicit Cayley graph, that this graph is disconnected with all components isomorphic, and that the resulting component is a 5-cube in dimension 20 and a bipartite graph in dimension 21. In both cases the maximum is exactly 2048, the value Cohn and Li already achieve, so their choice is maximal and the construction cannot be improved in those dimensions by enlarging the added-vector code. In dimension 19 the component is neither a cube nor bipartite; we give a Delsarte bound of 1536, improving the ratio bound of 1566, so that the construction yields $\tau_{19} \leq 12204$, and we conjecture that Ho's value 1280 is optimal. All structure is traced to the Steiner system $S(5, 8, 24)$, and the certificates are small enough to check by hand.

Keywords: kissing number, Golay code, Steiner system, independence number, Cayley graph, sphere packing.

2020 Mathematics Subject Classification: Primary 52C17; Secondary 05B05, 94B25, 05C69.

1 The construction and the question

Cohn and Li [1] prove that the kissing numbers in dimensions 17 through 21 are at least 5730, 7654, 11692, 19448 and 29768. For $n \in \{19, 20, 21\}$ their configuration is a union of three families of vectors, all of squared norm 8:

(i) $4\binom{n}{2}$ *root vectors*, having two entries equal to ± 2;

(ii) $2^7|C_n|$ *odd-sign vectors*, having eight entries equal to ± 1 supported on the weight-8 words of a code C_n, with an odd number of -1's;

(iii) the *added vectors* $\sqrt{8/n}\,s$ with $s \in \{\pm 1\}^n$, one for each word of a binary code that we call the *added-vector code* and denote A_n.

The first two families are fixed by the construction, and the bound produced is

$$\tau_n \;\geq\; 4\binom{n}{2} + 2^7|C_n| + |A_n|. \tag{1}$$

With $|C_{19}| = 78$, $|C_{20}| = 130$, $|C_{21}| = 210$ the first two terms contribute 10668, 17400 and 27720, and Cohn and Li take $|A_n|$ equal to 1024, 2048 and 2048 respectively.

Cohn and Li prove a maximality statement only in dimension 17 (their Lemma 3.1, that a subcode of C_{10} with minimal distance 6 has at most 192 words), and they write that "it is unclear to us what the limits of these sorts of constructions might be." Ho [4] answered this in dimension 19, replacing the linear code of 1024 words by a nonlinear code of 1280 words and raising the bound to 11948. Dimensions 20 and 21 have not been revisited. We settle them, and we bound dimension 19.

2 Reduction to an independence number

Fix $n \in \{19, 20, 21\}$ and write $d_n = \lceil n/4 \rceil$, so $d_{19} = d_{20} = 5$ and $d_{21} = 6$. Identify $s \in \{\pm 1\}^n$ with $c \in \mathbb{F}_2^n$ via $s_i = (-1)^{c_i}$.

Proposition 1. *Let D_n denote the extended binary Golay code punctured at a set P of $24-n$ coordinates, a code of length n with $|D_n| = 2^{12} = 4096$, and put*

$$W_n \;=\; \{\, w \in D_n \;:\; 0 < \mathrm{wt}(w) < d_n \,\}.$$

Then the admissible added-vector codes are exactly the subsets of D_n no two of whose elements differ by an element of W_n; equivalently

$$\max|A_n| \;=\; \alpha\big(\mathrm{Cay}(D_n, W_n)\big), \tag{2}$$

the independence number of the Cayley graph on D_n with connection set W_n.

Proof. All vectors of the configuration have squared norm 8, so for two of them the kissing condition $|u-v|^2 \geq 8$ is equivalent to $\langle u, v\rangle \leq 4$.

For a root vector $v = \pm 2e_i \pm 2e_j$ we get $\langle v, \sqrt{8/n}\,s\rangle \in \sqrt{8/n}\,\{0, \pm 4\}$, and $4\sqrt{8/n} \leq 4$ for $n \geq 8$; root vectors impose no condition.

Let v be an odd-sign vector with support S, $|S| = 8$, and signs σ. Then $\langle v, \sqrt{8/n}\,s\rangle = \sqrt{8/n}\,m$ with $m = \sum_{i\in S}\sigma_i s_i$ an even integer in $[-8, 8]$. The requirement $\sqrt{8/n}\,m \leq 4$ reads $m \leq \sqrt{2n}$, and $\sqrt{2n} < 8$ for $n \leq 21$ while $\sqrt{2n} > 6$ for $n \geq 19$; so the condition is exactly

$m \neq 8$, that is, $s|_S \neq \sigma$. The family (ii) contains all 2^7 sign patterns on S with an odd number of -1's, and that set is closed under $\sigma \mapsto -\sigma$ because 8 is even. Hence the condition over all odd-sign vectors with support S is that $s|_S$ have an *even* number of -1's, i.e. $\sum_{i \in S} c_i = 0$, i.e. $c \perp \mathbf{1}_S$. Therefore c must lie in $C_n^\perp$, where C_n is spanned by the weight-8 supports occurring in (ii). Those supports span the extended Golay code shortened at P; as the extended Golay code is self-dual and shortening is dual to puncturing, $C_n^\perp = D_n$.

Finally, for two added vectors, $\langle \sqrt{8/n}\, s, \sqrt{8/n}\, t\rangle = (8/n)\langle s, t\rangle$ and $\langle s, t\rangle = n - 2\,\mathrm{wt}(c + c')$, so the condition $\langle u, v\rangle \leq 4$ is $\mathrm{wt}(c + c') \geq n/4$, i.e. $\mathrm{wt}(c + c') \geq d_n$ since weights are integers. Thus $A_n \subseteq D_n$ avoids all differences in W_n, which is precisely independence in $\mathrm{Cay}(D_n, W_n)$. □

Remark 2. Because the automorphism group M_{24} of the extended Golay code is 5-transitive, the code D_n does not depend on the choice of P up to a permutation of coordinates, so (2) is well posed.

3 Component decomposition

The graphs $\mathrm{Cay}(D_n, W_n)$ are disconnected, and this reduces every question about them to a smaller graph.

Lemma 3. *Let G be a finite abelian group and $W \subseteq G \setminus \{0\}$ with $W = -W$, and set $H = \langle W\rangle$. Then the connected components of* $\mathrm{Cay}(G, W)$ *are the cosets of H, each inducing a graph isomorphic to* $\mathrm{Cay}(H, W)$*, and*

$$\alpha\big(\mathrm{Cay}(G, W)\big) = [G : H] \cdot \alpha\big(\mathrm{Cay}(H, W)\big).$$

Proof. From g one reaches exactly the elements $g + w_1 + \cdots + w_r$ with $w_i \in W$, that is, the coset $g + H$; and translation by g is an isomorphism $\mathrm{Cay}(H, W) \to \mathrm{Cay}(g + H, W)$. A set is independent in a disjoint union exactly when its trace on each component is independent, so the independence numbers add. □

Computing $H_n = \langle W_n\rangle$ in each case gives

n	$\dim H_n$	$\lvert H_n\rvert$	$[D_n : H_n]$
19	10	1024	4
20	5	32	128
21	10	1024	4

so that $\alpha(\mathrm{Cay}(D_{19}, W_{19})) = 4\,\alpha(\mathrm{Cay}(H_{19}, W_{19}))$, and similarly for $n = 21$, while dimension 20 splits into 128 copies of a graph on only 32 vertices.

4 The structure of W_n

The 759 weight-8 words of the extended Golay code (the *octads*) form the Steiner system $S(5, 8, 24)$: every 5-set of coordinates lies in exactly one octad [2, Ch. 10], [5, Ch. 20]. Writing

λ_i for the number of octads containing a fixed i-set, the recursion $\lambda_i = \frac{24-i}{8-i}\lambda_{i+1}$ with $\lambda_5 = 1$ gives

$$\lambda_5 = 1, \quad \lambda_4 = 5, \quad \lambda_3 = 21, \quad \lambda_2 = 77, \quad \lambda_1 = 253, \quad \lambda_0 = 759.$$

Two distinct octads meet in 0, 2 or 4 points [2, Ch. 10]. Puncturing at P with $|P| = p$ sends an octad O to a word of weight $8 - |O \cap P|$, so the words of D_n of weight less than d_n come from octads meeting P in nearly all of P.

Proposition 4. *Let $p = 24 - n$.*

(a) For $n = 21$ ($p = 3$), W_{21} consists of $\lambda_3 = 21$ words, each of weight 5.

(b) For $n = 20$ ($p = 4$), W_{20} consists of $\lambda_4 = 5$ words, each of weight 4, and these five words are pairwise disjoint; they partition the 20 coordinates into five blocks of four.

(c) For $n = 19$ ($p = 5$), W_{19} consists of $\lambda_5 = 1$ word of weight 3 together with $5(\lambda_4 - 1) = 20$ words of weight 4.

Proof. (a) Here $d_{21} = 6$, so we need the words of weight at most 5. A punctured octad has weight $8 - |O \cap P| \geq 5$, with equality exactly when $P \subseteq O$; there are $\lambda_3 = 21$ such octads, and no word of D_{21} has smaller weight.

(b) Here $d_{20} = 5$, so we need weight at most 4, forcing $P \subseteq O$; there are $\lambda_4 = 5$ such octads. If $O \neq O'$ both contain the 4-set P then $|O \cap O'| \geq 4$, hence $= 4$, so $O \cap O' = P$ and the punctured images $O \setminus P$, $O' \setminus P$ are disjoint. Five disjoint 4-sets inside 20 coordinates exhaust them.

(c) Here $d_{19} = 5$. Weight 3 requires $P \subseteq O$, and $\lambda_5 = 1$. Weight 4 requires $|O \cap P| = 4$: choose the omitted point of P in 5 ways, and for each of the 5 resulting 4-subsets there are $\lambda_4 = 5$ octads containing it, one of which contains all of P; this leaves $5 \cdot 4 = 20$ octads. □

5 Dimension 20: a disjoint union of 5-cubes

Theorem 5. $\mathrm{Cay}(D_{20}, W_{20})$ *is a disjoint union of* 128 *copies of the* 5*-dimensional hypercube* Q_5*, and* $\alpha(\mathrm{Cay}(D_{20}, W_{20})) = 128 \cdot 16 = 2048$*.*

Proof. By Proposition 4(b) the five elements of W_{20} have pairwise disjoint supports, so any nontrivial $\mathbb{F}_2$-combination of them has nonempty support: they are linearly independent. Hence $H_{20} = \langle W_{20} \rangle \cong \mathbb{F}_2^5$ with W_{20} a basis, and $\mathrm{Cay}(H_{20}, W_{20})$ is the Cayley graph of $\mathbb{F}_2^5$ with respect to a basis, that is, the 5-cube Q_5. Its independence number is $2^4 = 16$, attained by the even-weight vertices. Since $[D_{20} : H_{20}] = 4096/32 = 128$, Lemma 3 gives $\alpha = 128 \cdot 16 = 2048$. □

6 Dimension 21: a parity certificate

Lemma 6. *Let Γ be a k-regular bipartite graph on N vertices, $k \geq 1$, whose two parts have $N/2$ vertices each. Then $\alpha(\Gamma) = N/2$.*

Proof. Each part is independent, so $\alpha \geq N/2$. A k-regular bipartite graph satisfies Hall's condition and therefore has a perfect matching, of $N/2$ edges; an independent set contains at most one endpoint of each matching edge, so $\alpha \leq N/2$. □

Theorem 7. *Let $\varphi(c) = \sum_{i=1}^{21} c_i$ be the total parity functional. Then $\varphi(w) = 1$ for every $w \in W_{21}$; consequently $\mathrm{Cay}(D_{21}, W_{21})$ is bipartite with parts of size 2048 and $\alpha(\mathrm{Cay}(D_{21}, W_{21})) = 2048$.*

Proof. Every $w \in W_{21}$ has weight 5 by Proposition 4(a), so $\varphi(w) = \mathrm{wt}(w) \bmod 2 = 1$. Since φ takes the value 1 on D_{21}, its kernel meets D_{21} in a subgroup of index 2, splitting D_{21} into two sets of 2048 words. Every edge joins c to $c + w$ with $w \in W_{21}$, and $\varphi(c + w) = \varphi(c) + 1$, so every edge crosses the partition. The graph is 21-regular, so Lemma 6 applies. □

Remark 8. The same argument applies in dimension 20 if one prefers a uniform treatment: the five blocks of Proposition 4(b) admit a transversal T, and $\varphi_{20}(c) = \sum_{i \in T} c_i$ satisfies $\varphi_{20}(w) = 1$ for each $w \in W_{20}$. Theorem 5 is sharper in that it identifies the components exactly.

Corollary 9. *In the odd-sign construction the added-vector code satisfies $\max|A_{20}| = \max|A_{21}| = 2048$. Since Cohn and Li already take $|A_{20}| = |A_{21}| = 2048$, their choice is maximal: the bounds $\tau_{20} \geq 19448$ and $\tau_{21} \geq 29768$ cannot be improved by enlarging the added-vector code.*

7 Dimension 19: bounds

By Proposition 4(c) the set W_{19} contains one word of weight 3 and twenty of weight 4: the weights have *mixed* parity, so total parity does not separate. In fact no linear functional does.

Proposition 10. *There is no linear $\varphi\colon \mathbb{F}_2^{19} \to \mathbb{F}_2$ with $\varphi(w) = 1$ for all $w \in W_{19}$, and $\mathrm{Cay}(D_{19}, W_{19})$ is not bipartite.*

Proof. The system $\varphi(w) = 1$, $w \in W_{19}$, is a linear system over $\mathbb{F}_2$; row reduction shows it is inconsistent, some $\mathbb{F}_2$-combination of the w's vanishing while an odd number of them is used. Equivalently the graph contains an odd closed walk. □

Thus the obstruction that pins $|A_n|$ to 2048 in dimensions 20 and 21 is absent in dimension 19, which is precisely why Ho's improvement was possible there. By Lemma 3 it suffices to study the component $\mathrm{Cay}(H_{19}, W_{19})$, a 21-regular graph on 1024 vertices; Ho's code gives $\alpha(\mathrm{Cay}(H_{19}, W_{19})) \geq 320$.

For an upper bound, let A be independent in $\mathrm{Cay}(H, W)$ with $H \cong \mathbb{F}_2^{10}$ and put $a_g = |A|^{-1}\#\{(x, y) \in A^2 : x + y = g\}$. Then $a_0 = 1$, $a_g \geq 0$, $a_g = 0$ for $g \in W$, and for every character χ_u,

$$\sum_g a_g\, \chi_u(g) = |A|^{-1}\Big|\sum_{x \in A} \chi_u(x)\Big|^2 \;\geq\; 0.$$

Maximizing $\sum_g a_g$ subject to these constraints is the Delsarte linear programming bound [3] for this translation scheme, a linear program in 1024 variables whose optimum bounds $|A|$ from above. Its value is exactly 384.

Theorem 11. $320 \le \alpha(\mathrm{Cay}(H_{19}, W_{19})) \le 384$*, hence* $1280 \le \alpha(\mathrm{Cay}(D_{19}, W_{19})) \le 1536$*, and the odd-sign construction yields*

$$11948 \;\le\; \tau_{19}^{\mathrm{osc}} \;\le\; 12204,$$

where τ_{19}^{osc} *denotes the best bound obtainable from* (1) *in dimension 19.*

The upper bound 1536 improves the ratio (Hoffman) bound, which gives 391 on the component and hence 1566. It is also the limit of this family of relaxations: since $\mathrm{Cay}(H, W)$ is a Cayley graph on an abelian group, an optimal solution of the corresponding semidefinite relaxation may be taken invariant under translation, and translation-invariant matrices are simultaneously diagonalized by the characters, so the semidefinite relaxation reduces to the linear program above. Adjoining the valid inequalities $a_g \le 1$ does not change the optimum.

Conjecture 12. $\alpha(\mathrm{Cay}(H_{19}, W_{19})) = 320$*; equivalently* $\alpha(\mathrm{Cay}(D_{19}, W_{19})) = 1280$ *and Ho's added-vector code is maximal, so that the odd-sign construction is maximal in dimensions 19, 20 and 21 alike.*

In support of the conjecture, iterated local search on the 1024-vertex component, with 1602 independent restarts of $2 \cdot 10^5$ steps each, reached 320 repeatedly and never exceeded it; a search over $3.7 \cdot 10^5$ subgroups $K \le H_{19}$ with $K \cap W_{19} = \emptyset$, lifting maximum independent sets of the quotients, produced nothing larger. The spectrum of $\mathrm{Cay}(H_{19}, W_{19})$ is

$$21^1,\ 19^1,\ 11^{16},\ 9^{16},\ 5^{130},\ 3^{290},\ 1^{160},\ (-3)^{120},\ (-5)^{200},\ (-7)^{80},\ (-11)^5,\ (-13)^5,$$

so the graph is not strongly regular and no standard family applies.

8 Scope

Corollary 9 and Theorem 11 concern the odd-sign construction, not the kissing number. They say that within that construction, holding the root vectors and odd-sign vectors fixed as Cohn and Li do, the added-vector code cannot be enlarged in dimensions 20 and 21, and is confined to $[1280, 1536]$ in dimension 19. They do not assert that $\tau_{20} = 19448$ or $\tau_{21} = 29768$; both remain open, and improvements from other constructions are not excluded. Indeed Cohn and Li expect their configurations not to be optimal.

The ingredients are standard: the Steiner system $S(5, 8, 24)$ and its intersection numbers, the duality of shortening and puncturing, the existence of a perfect matching in a regular bipartite graph, and the Delsarte linear programming bound. What the note contributes is the reduction (2), the component decomposition of Lemma 3, the identification of the dimension-20 components as 5-cubes, the parity certificate in dimension 21, the resulting maximality statement answering a question the authors of the construction explicitly left open, and the improved bound in dimension 19.

Changes from Version 1

Version 1 proved Corollary 9 by the bipartiteness argument alone. Version 2 adds the component decomposition (Lemma 3), the sharper description of dimension 20 as a disjoint union of 5-cubes (Theorem 5), and Section 7 on dimension 19, with the Delsarte bound 1536 and Conjecture 12.

References

[1] H. Cohn and Y. Li, Improved kissing numbers in seventeen through twenty-one dimensions, preprint, 2024. Available at `https://arxiv.org/abs/2411.04916`.

[2] J. H. Conway and N. J. A. Sloane, *Sphere Packings, Lattices and Groups*, 3rd ed., Springer, 1999.

[3] P. Delsarte, An algebraic approach to the association schemes of coding theory, *Philips Res. Rep. Suppl.* **10** (1973).

[4] B. S. Ho, A new lower bound for the kissing number in 19 dimensions, preprint, 2026.

[5] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, 1977.