A certified opaque barrier for the unit disc of length 4.79984937…

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> Version 2 (July 2026). The historical attribution in the Abstract and §1 is revised: the upper bound traces to Makai (1980), with its origin in Fejes Tóth (1974). The certified value, the opacity proof, and the code are unchanged from version 1. This markdown is the source of the deposited PDF (built by papers/build_pdf.py); the code named below reproduces every number.

Abstract

An opaque set for the unit disc is a set meeting every straight line that meets the disc; the infimum of the lengths of such sets is the beam detection constant, whose value is unknown. The best known upper bound is a three-piece barrier due to E. Makai Jr. (1980), quoted in the literature as approximately 4.7998; the sharpest published evaluation of the construction that can be checked against a primary source, by Faber and Mycielski (1986), gives 4.79989 at rounded parameters. I evaluate the same three-piece family at its true stationary point and certify an opaque barrier of length 4.799849374678549857544655408276…, and prove its opacity. This improves the Faber–Mycielski evaluation by a certified margin exceeding 4×10^{-5} and fixes the constant of the construction to thirty digits. Makai's own value is quoted only to five significant digits, with which the value here agrees; whether the stationary-point evaluation improves or rediscovers his original figure cannot be determined from sources accessible to me, and no priority over Makai is claimed. The opacity proof given here is exact: it rests on a closed-form apex identity and a convexity argument, and requires no numerical sweep over directions. Its only numerical inputs are two chord parameters, certified by interval arithmetic to lie in (0, 1). The length is likewise a certified interval enclosure of width below 10^{-59}. I claim no new construction and no optimality for the beam detection constant itself.

1. The problem

A set $B \subseteq \mathbb{R}^2$ is opaque for a convex body K if B meets every line that meets K. For the unit disc D the infimum of the lengths of opaque sets is the beam detection constant (also the beamsplitter, or trench-digger's, constant). It satisfies

$$\pi \le L \le 4.7998,$$

the lower bound essentially due to Jones [4], recently sharpened by Kiderlen and Pausinger [5] to $L \ge \pi + 1.076\times10^{-6}$. The problem descends from Tarski's plank problem: Fejes Tóth [9] posed the shortest-barrier ("skeleton") question in 1974 and reported early disconnected constructions of Makai, the simplest of length $\pi + \sqrt{3} = 4.8732…$. Makai's 1980 paper [6] is cited by the modern literature [5] as the source of the three-piece upper bound, quoted there to five significant digits as ≈ 4.7998 and later rediscovered by J. Day (see [8]). The sharpest evaluation of the construction that I can check against a primary source is by Faber and Mycielski [2], whose Fig. 9(c) gives the fraction .763926466 of the boundary — that is, 4.799891547 — at the rounded gap angles (.96, 1.04, .7, 1.2); Finch [3, §8.11] catalogues the figure 4.79989. None of this should be confused with the two-piece barrier of Faber, Mycielski and Pedersen [1, §6], who prove that the shortest closed opaque set for the circle having two components has length 4.818924…. The exact constant is open and the gap is wide; recent progress has been on the lower bound.

Two contributions. First, the checkable published evaluations correspond to the three-piece family at parameters that are not stationary for the length functional; moving to the true stationary point lowers the Faber–Mycielski figure by 4.2×10^{-5} (§2). Whether it also improves Makai's original value cannot be decided from accessible sources, as explained above. Second, I give a proof of opacity for this barrier that is exact and self-contained (§3), replacing direction sweeps by a closed-form argument.

2. The barrier

Work in the frame in which the uncovered part of the circle is centred on the positive x-axis. The barrier is determined by four consecutive gap angles $t_1,t_2,t_3,t_4 \in (0, \pi)$ with $h = \frac{1}{2}(t_1+t_2+t_3+t_4) < \pi$. Put $\varphi_0 = -h$ and $\varphi_k = \varphi_{k-1} + t_k$, so $\varphi_4 = h$ and the four gap intervals $[\varphi_{k-1}, \varphi_k]$ tile $[-h, h]$ exactly. Write $P(\varphi) = (\cos\varphi, \sin\varphi)$. The apex of gap k is $E_k = \sec(t_k/2)\cdot(\cos m_k, \sin m_k)$ with $m_k = (\varphi_{k-1}+\varphi_k)/2$, so $|E_k| = \sec(t_k/2)$ exactly (identity (1)). The barrier B is the covered arc $A = \{P(\varphi) : \varphi \in [h, 2\pi-h]\}$ together with two bow tangent segments and two arrow segments R_1, R_2, each arrow being an altitude foot F onto a chord of the apices; the connected piece $C = A \cup$ (bows). The length functional L is the arc angle plus the two bow tangents plus the two altitude lengths. [beam_3arc.py]

The published figure 4.79989 is reproduced to all five quoted digits by the parameters (0.96, 1.04, 0.7, 1.2), at which L = 4.79989154701995156793698878787164…. These parameters are not stationary. The stationary point of the length functional is

$$(t_1,t_2,t_3,t_4) = (0.98156421634813442669,\ 1.04081529919525904389,\ 0.67398291116704119189,\ 1.20424629367152261166). \quad (3)$$

Numerically the gradient there has norm below 10^{-21}; the Hessian, computed numerically, has eigenvalues ≈ 0.0647, 0.2982, 0.3578, 0.8854, all positive, so the point is numerically a strict local minimum. (This is stated for orientation only; it enters neither the opacity proof nor the certified length.) At (3), outward-rounded interval arithmetic gives the certified enclosure

$$4.79984937467854985754465540827591002737\,02 < L_3 < 4.79984937467854985754465540827591002737\,03, \quad (4)$$

of width below 1.2×10^{-59}, so

$$\mathbf{L_3 = 4.7998493746785498575446554082 76\ldots}$$

No binary floating point enters this enclosure. In particular $L_3 < 4.79989$ with certified margin at least 4.0625×10^{-5}, and L_3 falls below the rounded-parameter value by at least 4.2172×10^{-5}.

3. Opacity

I prove opacity exactly. The argument does not sweep over directions and does not require a separate limiting argument for tangent lines. Write $u(\theta) = (\cos\theta, \sin\theta)$ and, for a compact set S, let $\Sigma_S(\theta) = \{ s\cdot u(\theta) : s \in S \}$ be the shadow of S in direction θ.

Lemma 3.1 (Reduction). B is opaque for the unit disc iff $\Sigma\_B(\theta) \geq$(set) $[-1, 1]$ for every θ. (A line $\ell\_\{\theta,p\} = \{x : x\cdot u(\theta) = p\}$ meets the disc iff $|p| \leq 1$ and meets B iff $p \in \Sigma\_B(\theta)$.)

Since C, R_1, R_2 are each compact and connected, their shadows are compact intervals. Write $M(\theta) = \max\_\{b\in B\}\ b\cdot u(\theta)$, $m(\theta) = \min\_\{b\in B\}\ b\cdot u(\theta)$.

Lemma 3.2 (The shadow is one interval). If foot F_1 lies on $[E_1,E_4]$ and F_2 on $[E_2,E_4]$, then $\Sigma\_B(\theta) = \Sigma\_C(\theta) \cup \Sigma\_\{R_1\}(\theta) \cup \Sigma\_\{R_2\}(\theta) = [m(\theta), M(\theta)]$ for every θ.
($F_1 = (1-s_1)E_1 + s_1E_4$ projects between E_1 and E_4, both in C, so into Σ_C; and $F_1 \in R_1$, so Σ_C and $\Sigma\_\{R_1\}$ meet. Then F_2 projects between $E_2 \in R_1$ and $E_4 \in C$, hence into that union, and $F_2 \in R_2$, chaining $\Sigma\_\{R_2\}$. The union of overlapping compact intervals is one interval.)

Lemma 3.3 (Apex reach). $M(\theta) \geq 1$ for every θ.
(For θ in the arc range, $P(\theta) \in A \subseteq B$ gives $P(\theta)\cdot u = 1$. For $\theta \in [-h, h]$, the gaps tile, so $\theta \in [\varphi\_\{k-1\}, \varphi\_k]$ and $|m\_k - \theta| \leq t\_k/2 < \pi/2$; then $E\_k\cdot u(\theta) = \sec(t\_k/2)\cos(m\_k-\theta) \geq \sec(t\_k/2)\cos(t\_k/2) = 1$, and each apex is on the barrier.)

Theorem 3.4. For t as in (3), B(t) is opaque, and $L \leq L_3 = 4.799849374678\ldots < 4.79989$.
*(Interval arithmetic certifies $s_1 = 0.294660041656711\ldots$, $s_2 = 0.467335706033416\ldots$, each strictly in (0,1), so Lemma 3.2 gives $\Sigma\_B(\theta) = [m(\theta), M(\theta)]$. By Lemma 3.3, $M(\theta) \geq 1$, and applying it at θ+π with $u(\theta+\pi) = -u(\theta)$ gives $m(\theta) = -M(\theta+\pi) \leq -1$. Hence $\Sigma\_B(\theta) \geq$(set) $[-1,1]$
for every θ, and B is opaque by Lemma 3.1.)*

Remark 3.5. Lemma 3.1 is an equivalence on the closed interval $[-1, 1]$, so tangent lines ($|p| = 1$) are covered directly; no separate limiting argument is needed. And no branch-and-bound over θ is required: Lemma 3.3 is a closed-form inequality valid for all θ.

Remark 3.6. The only numerical inputs to Theorem 3.4 are the two scalars s_1, s_2, which enter solely through the qualitative statement $s_1, s_2 \in (0,1)$ and are certified with enormous margin. Identity (1) and Lemmas 3.1–3.3 are exact. The length (4) is a certified enclosure and plays no role in the opacity proof.

Independent cross-check. A direct per-line test — a separate code path from the shadow argument — examined 9,003,000 lines and found none unblocked. This is float, not part of the proof. [hunt/opacity.py]

4. Reproducibility

Length: verify-upper/beam_3arc.py. Opacity (shadow bounds, apex identity, foot-on-chord): verify-upper/opacity_shadow_certified.py. Independent per-line test and general engine: hunt/opacity.py. Each prints its certified enclosures and verdict.

References

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