Certified upper bounds for Fejes Tóth's point-goalie problem at n = 4 and 5

Vince Gonzalez · ORCID 0009-0005-3640-014X

Abstract

In 1974 László Fejes Tóth posed the following problem: place n points in the plane so as to minimise the largest distance from a line meeting the unit-radius disc to the nearest point. Writing r_n for the optimum, he proved $r_1 = r_2 = 1$ and $r_3 = 3/5$ exactly, gave constructions showing $r_4 \le 0.471\ldots$, $r_5 \le 0.406\ldots$ and $r_6 \le 1/3$, and wrote that r_n is unknown for n > 3. The problem was later named the point goalie problem; its modern literature treats the asymptotic and density regimes, and I am not aware of any published improvement of the finite-n values. This note certifies, in exact rational interval arithmetic,

$$r_4 \le 0.468672 \qquad r_5 \le 0.394954,$$

improving the 1974 bounds by 0.0023 and 0.0106 respectively. The certifying configurations are given by exact rational coordinates, and the certifier is validated by negative controls against Fejes Tóth's proven value $r_3 = 3/5$. For n = 6 an unstructured search converged back to Fejes Tóth's own configuration and produced no improvement; that is reported as a negative result, not as evidence of optimality. Finally, Fejes Tóth's skeleton argument applied to the certified opaque barrier of length 4.799849374678… (Zenodo 10.5281/zenodo.21701081) gives $\limsup n \cdot r_n \le 2.39992468\ldots$, improving the asymptotic upper bound $\frac{1}{2}(\pi + \sqrt{3}) = 2.43682\ldots$ stated in his paper; the best known asymptotic lower bound, due to Richardson and Shepp, is 1.001. No optimality is claimed for anything presented here.

1. The problem

Fejes Tóth [1] formulated the problem as the point-line dual of his reformulation of Tarski's plank problem: distribute n points in the plane of a convex domain T so as to minimise the maximum distance between a line intersecting T and the point nearest to it. For T the disc of unit radius he proved that $r_1 = r_2 = 1$ and that $r_3 = 3/5$, the extremal triple being the vertices of an equilateral triangle circumscribed about a concentric circle of radius 2/5. For larger n he gave constructions: a quadrilateral yielding $r_4 \le 0.471\ldots$, a pentagon determined by the cubic $(1+b)^3 = 4(1-b)$ yielding $r_5 \le 0.406\ldots$ (the value of that construction evaluates to 0.40552…), and, for n = 6, a regular pentagon circumscribed about a concentric circle of radius 2/3 together with the centre, yielding $r_6 \le 1/3$ — with the observation that a whole family of configurations attains the same value. He wrote that determining r_n for n > 3 appears hopeless and that even the asymptotics are difficult.

The problem did not disappear. Erdős and Pach [2] solved a whole-plane density version. Richardson and Shepp [3] studied the disc problem in the asymptotic regime under the name that stuck — the point goalie problem — casting it as an infinite integer program whose linear relaxation has optimal weight $1/\varepsilon$, and proving the integer version strictly larger: at least $1.001/\varepsilon$. Their paper contains no finite-n results, and its bibliography reaches Fejes Tóth's problem only through [2]; the 1974 paper itself is not cited there. The maximum-norm sibling (covering all secants of a square) was studied by Bárány and Füredi [4], Kern and Wanka [5], and Valtr [6]. Dumitrescu, Jiang and Pach [7] name the lineage in their survey of opaque sets: discrete barriers of few points "arrive at a problem raised by László Fejes Tóth", "later coined suggestively as the 'point goalie problem'". On the sphere, where polarity makes the point and line versions equivalent, the problem is completely solved as the zone conjecture, proved by Jiang and Polyanskii [8]; see Zhao [9] for the congruent case. For the related minimum-length questions: Richardson and Shepp recount that Eggleston [10] proved the shortest connected set meeting every line through the disc to have length $\pi + 2$, and the best known unrestricted barrier is the three-piece construction whose certified length is 4.799849374678… [11].

Against this literature, the exact finite-n values appear untouched since 1974. That claim is scoped precisely: it rests on reading [1] and [3] in full, on the searches documented in §5, and it is a statement about what was found, not a proof of absence.

2. Results

Theorem. Let X_4 and X_5 be the point configurations with the exact rational coordinates below. Every line meeting the closed disc of radius 1 passes within distance 0.468672 of a point of X_4, and within 0.394954 of a point of X_5. Hence

$$r_4 \le 0.468672, \qquad r_5 \le 0.394954.$$

X_4:

```
(−563701/1000000, −97227/500000)      (136301/1000000, −870417/1000000)
(−21447/50000,     48137/62500)       (740247/1000000,  126993/500000)
```

X_5:

```
(−19741/31250,   −48683/250000)       (−541749/1000000, 290597/500000)
(159987/250000,   634717/1000000)     (587553/1000000, −4281/12500)
(−74451/1000000, −430277/500000)
```

The configurations were found by unstructured numerical search and are certified as they stand; no optimality is claimed for them, and the true values of r_4 and r_5 are presumably smaller. For comparison, Fejes Tóth's constructions give 0.471… and 0.40552…, so the improvements are approximately 0.0023 and 0.0106.

A negative result for n = 6. The same search, run for six points from random starts, converged to Fejes Tóth's own configuration — five points at the vertices of a regular pentagon circumscribed about a concentric circle of radius 2/3 (circumradius 0.8240…), and one point at the centre — reaching 0.333415 on the evaluation grid, slightly above 1/3, without improving it. This corroborates his configuration and is consistent with his remark that a family of configurations attains 1/3; it is not evidence that $r_6 = 1/3$.

3. The certificate

A line with unit normal $u(\theta)$ and signed offset p meets the disc iff $|p| \le 1$, and its distance to a point x is $|\langle x, u(\theta)\rangle - p|$. The claimed bound r therefore certifies iff for every θ the intervals $[q_i(\theta) - r, q_i(\theta) + r]$ around the projections $q_i(\theta) = \langle x_i, u(\theta)\rangle$ cover $[-1, 1]$ — the same one-parameter covering reduction used throughout the certified-barrier work [11].

The certifier subdivides θ adaptively. On a θ-box, each projection is enclosed in an interval q_i by outward-rounded arithmetic (exact rational endpoints; sine and cosine by validated enclosures), and the zone certain to be covered for every θ in the box is $[q_i.\mathrm{hi} - r, q_i.\mathrm{lo} + r]$. A greedy sweep checks whether these guaranteed zones cover $[-1, 1]$; where they do not, the box is bisected. Termination with all boxes covered proves the bound for all θ. The arithmetic kernel is the one described, with its soundness history, in [11].

Two disciplines deserve record. First, negative controls: on Fejes Tóth's proven-optimal triangle (snapped to rationals at 10^{-6}) the certifier refuses $r = 0.599990$ and accepts $r = 0.600010$, bracketing his exact 3/5 — a certifier that cannot reject a false claim proves nothing by accepting a true one. Second, the quoted bounds were obtained by bisecting r against the certifier itself, because a first attempt exposed an error worth reporting: a floating-point maximum over a θ-grid underestimates the true critical distance, and targets set a few millionths above grid values failed to certify. No floating-point quantity remains on the claim path; floats served only to find the configurations, which are then certified as exact rational objects.

4. The asymptotic regime

Fejes Tóth observed that points distributed along a "skeleton" — in modern terms an opaque set: a set met by every line meeting the disc — inherit its blocking property, and he instantiated this with a skeleton of length $\pi + \sqrt{3}$ to obtain $\limsup n \cdot r_n \le \frac{1}{2}(\pi + \sqrt{3}) = 2.43682\ldots$. The argument is short enough to include. Let B be an opaque set for the disc consisting of m rectifiable curves of total length h. Along each curve place samples at arc-length spacing at most 2r, centred, so that every point of the curve lies within arc-distance r of a sample; this needs at most $h/(2r) + m$ points. A line meeting the disc meets B at some point p, and the sample nearest to p along its curve is at Euclidean distance at most its arc-distance, hence at most r. So n points suffice whenever $n \ge h/(2r) + m$, giving $r_n \le h/(2(n - m))$ and $\limsup n \cdot r_n \le h/2$.

Instantiating with the certified three-piece barrier of length 4.799849374678… [11] (m = 3):

$$\limsup n \cdot r_n \le 2.39992468\ldots,$$

improving the constant stated in [1]. In the other direction Richardson and Shepp [3] prove that asymptotically at least $1.001/\varepsilon$ points are needed, i.e. $\liminf n \cdot r_n \ge 1.001$, and write that the true answer is probably considerably larger. The asymptotic constant therefore lies in [1.001, 2.39992468…]; the finite-n values here give $4 \cdot r_4 \le 1.8747$ and $5 \cdot r_5 \le 1.9748$, consistent with a slow approach to the constant from below.

5. Scope

Everything claimed is an upper bound certified for a specific rational configuration; nothing is claimed optimal. The n = 6 outcome is a failure to improve, not a proof. The novelty claim — that the finite-n values have seen no published improvement since 1974 — is an awareness claim supported by: reading [1] in the Hungarian original; reading [3] in full (asymptotic only; [1] not cited there); the naming lineage in [7];

and searches across the opaque-set, plank-problem and goalie literatures, including the 2022–2023 survey articles accompanying the new edition of Fejes Tóth's Lagerungen, which discuss the plank problem without mentioning this dual. Two sources could not be checked in full: the printed notes of the 2023 Lagerungen edition and the problem book of Brass, Moser and Pach. Corrections are welcome and would be incorporated with attribution.

6. Reproducibility

hunt/ftpoints.py — the critical-distance evaluator (float; measurement only) with Fejes Tóth's r_3 and r_6 configurations as known-answer gates, both reproducing his values to nine digits on the evaluation grid. hunt/ftpoints_polish.py — the search and the rational snapping that produced X_4 and X_5. verify-upper/goalie_certified.py — the certifier, the negative controls, and the bisection of r; it prints every verdict and exits non-zero if any check fails, including the controls. core/exact.py — the arithmetic kernel. The certified-barrier constant and its provenance are in [11].

References

- [1] L. Fejes Tóth, Megjegyzések Tarski sávproblémájának egy duálisához (Remarks on a dual of Tarski's plank problem; in Hungarian), Matematikai Lapok 25 (1974) 13–20.
- [2] P. Erdős, J. Pach, On a problem of L. Fejes Tóth, Discrete Math. 30 (1980) 103–109.
- [3] T. Richardson, L. Shepp, The "Point" Goalie Problem, Discrete Comput. Geom. 30 (2003) 649–669. doi:10.1007/s00454-003-2713-3.
- [4] I. Bárány, Z. Füredi, Covering all secants of a square, in: Intuitive Geometry (Siófok 1985), Colloq. Math. Soc. János Bolyai 48, North-Holland (1987) 19–27.
- [5] W. Kern, A. Wanka, On a problem about covering lines by squares, Discrete Comput. Geom. 5 (1990) 77–82.
- [6] P. Valtr, Unit squares intersecting all secants of a square, Discrete Comput. Geom. 11 (1994) 235–239.
- [7] A. Dumitrescu, M. Jiang, J. Pach, Opaque sets, Algorithmica 69 (2014) 315–334; arXiv:1005.2218.
- [8] Z. Jiang, A. Polyanskii, Proof of László Fejes Tóth's zone conjecture, Geom. Funct. Anal. 27 (2017) 1367–1377.
- [9] Y. Zhao, Exploring a planet, revisited, arXiv:2110.04376 (2021).
- [10] H. G. Eggleston, *The maximal in-radius of the convex cover of a plane connected set of given length*, Proc. London Math. Soc. (3) 45 (1982) 456–478. (Cited here for the connected-goalie value π + 2, as recounted in [3].)
- [11] V. Gonzalez, *A certified opaque barrier for the unit disc of length 4.79984937…*, Zenodo (2026), version 2, doi:10.5281/zenodo.21701081.