

First-Return Walks on Vertex-Transitive Graphs

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Abstract

Let Γ be a connected vertex-transitive graph on N vertices with adjacency spectrum $\{(\lambda_i, m_i)\}$. A closed walk based at a vertex is indecomposable, or a first return, if it revisits its base only on the final step. We record the observation that the first-return counts are obtained from the spectrum alone by a single rational operation,

$$\sum I(L) x^L = 1 - 1/f(x), \text{ where } f(x) = (1/N) \sum_i m_i / (1 - \lambda_i x),$$

so that each graph contributes an integer sequence satisfying a linear recurrence whose order is the number of distinct nonzero eigenvalues. Specialising to the Hamming graphs $H(k, q)$ — the Cayley graphs of $(\mathbb{Z}/q)^k$ with the weight-one generators — gives a two-parameter array with an explicit closed form. The case $q = 2$ is the hypercube; its row for fixed k counts, under a dictionary we make precise, the irreducible theorems of the pure equivalential calculus in k variables. Three rows of the resulting table were absent from the OEIS and have been submitted: the 4-cube (A398262), the Petersen graph (A398309), and the icosahedron (A398310).

1. Scope

The two ingredients used here are classical: the expression of closed-walk counts as spectral moments, that is $W(L) = (1/N) \operatorname{tr}(A^L) = (1/N) \sum_i m_i \lambda_i^L$, immediate from the diagonalisation of the adjacency matrix and standard in algebraic graph theory (Godsil and Royle [3]); and the first-return decomposition of walks, equivalently the INVERT transform on ordinary generating functions (Flajolet and Sedgewick [2], §1.5; Bernstein and Sloane [1]). Neither is new, and no claim of new mathematics is made for Theorem 1. The spectral-moment identity is elementary and self-contained; we prove the form we use in §3.

The contribution of this note is threefold. First, the formula is stated uniformly, so that any vertex-transitive graph yields its first-return sequence by a single mechanical operation on its spectrum. Second, the Hamming specialisation gives a two-parameter closed form covering hypercubes, rook's graphs, and their q -ary generalisations in one expression. Third, we record the sequences the construction produces which were not in the OEIS, and we make explicit the dictionary between the hypercube rows and the pure equivalential calculus, which we have not found stated in the literature. The single-graph case $k = 3$, $q = 2$ already appears as A328778 (Critzler, 2019), whose formula field states the construction $2 - 1/f(x)$ for the cube; what follows is the general statement and its consequences.

2. First returns and the renewal identity

Let Γ be connected and regular, and fix a base vertex v . A closed walk of length L at v is a walk $v = x_0, x_1, \dots, x_L = v$. It is a first return if $x_t \neq v$ for $0 < t < L$. Let $W(L)$ count all closed walks of length L at v , with $W(0) = 1$, and let $I(L)$ count the first returns, with $I(0) = 0$.

Every closed walk factors uniquely as a concatenation of first returns: cut the walk at each intermediate visit to v . Hence, with $f(x) = \sum W(L) x^L$ and $c(x) = \sum I(L) x^L$,

$$f = 1/(1 - c), \text{ so } c = 1 - 1/f. \quad (1)$$

Equivalently $\{W(L)\}$ is the INVERT transform of $\{I(L)\}$. The form $2 - 1/f = 1 + c$ prepends the empty walk and is the normalisation used below.

3. The spectral formula

Theorem 1. Let Γ be a connected vertex-transitive graph on N vertices with adjacency eigenvalues λ_i of multiplicity m_i . Then

$$f(x) = (1/N) \sum_i m_i / (1 - \lambda_i x), \quad (2)$$

and the first-return generating function is $c(x) = 1 - 1/f(x)$. Writing $f = P/Q$ in lowest terms, $1 + c = (2P - Q)/P$, so $\{I(L)\}$ satisfies the linear recurrence whose characteristic polynomial is the reciprocal of P ; its order is the number of distinct eigenvalues with nonzero coefficient, and its growth rate is the reciprocal of the smallest-modulus root of P .

Proof. The (v,v) entry of A^L counts closed walks of length L at v . Vertex-transitivity makes this independent of v , so it equals $(1/N) \text{tr}(A^L) = (1/N) \sum_i m_i \lambda_i^L$. Summing $\sum (\cdot) x^L$ and using $\sum \lambda^L x^L = 1/(1 - \lambda x)$ gives (2). The renewal identity (1) gives c . Clearing denominators gives the rational form and hence the recurrence. ■

Corollary 2 (Cayley graphs). If $\Gamma = \text{Cay}(G, S)$ with S symmetric, the spectrum is determined by the irreducible representations of G . For abelian G the irreps are the characters χ , the eigenvalues are the character sums $\lambda_\chi = \sum \chi(s)$ each of multiplicity one, and $f(x) = (1/|G|) \sum_\chi 1/(1 - \lambda_\chi x)$.

4. The Hamming family

Let $H(k,q)$ be the Cayley graph of $(\mathbb{Z}/q)^k$ whose connection set is all vectors of Hamming weight one: the k -fold Cartesian power of K_q , with $N = q^k$ vertices, regular of degree $k(q-1)$. Its eigenvalues are $\lambda_t = k(q-1) - qt$, with multiplicity $m_t = C(k,t)(q-1)^t$, for $t = 0, \dots, k$.

Corollary 3. The first-return counts of $H(k,q)$ have generating function $1 - 1/f$, where

$$f(x) = q^{-k} \sum_{t=0..k} C(k,t)(q-1)^t / (1 - (k(q-1) - qt)x). \quad (4)$$

Two degenerate cases: $q = 2$ gives the hypercube Q_k (bipartite, so $I(L) = 0$ for odd L), and $k = 1$ gives K_q , with $I(L) = (q-1)(q-2)^{(L-2)}$ for $L \geq 2$.

Small rows, computed exactly from (4) and verified three independent ways (§7):

graph	I(1), I(2), ...	recurrence	OEIS
$H(3,2) = Q_3$	3, 0, 12, 0, 84, 0, 588, ...	$a(n) = 7 a(n-1), n > 2$	A328778
$H(4,2) = Q_4$	4, 0, 24, 0, 288, 0, 4032, ...	$a(n) = 16 a(n-1) - 24 a(n-2)$	A398262
$H(5,2) = Q_5$	5, 0, 40, 0, 680, 0, 14440, ...	—	—
$H(2,3)$	4, 4, 20, 68, 244, 868, ...	$a(n) = 3 a(n-1) + 2 a(n-2)$	—
$H(3,3)$	6, 6, 54, 198, 1206, 6534, ...	$a(n) = 6 a(n-1) + 3 a(n-2) - 24 a(n-3)$	—
$H(4,3)$	8, 8, 104, 392, 3368, ...	$10 a(n-1) - 3 a(n-2) - 118 a(n-3) + 88 a(n-4)$	—
$H(2,4)$	6, 12, 60, 336, 1896, ...	$a(n) = 6 a(n-1) - 2 a(n-2)$	—
$H(3,4)$	9, 18, 144, 936, 7164, ...	$a(n) = 12 a(n-1) - 23 a(n-2) - 42 a(n-3)$	—

For instance the generating function of $H(2,3)$ is $(4x^2 - 8x^3)/(1 - 3x - 2x^2)$ and that of $H(2,4)$ is $(6x^2 - 24x^3)/(1 - 6x + 2x^2)$. Note $H(2,3)$ is the 3×3 rook's graph, isomorphic to the Paley graph of order 9, and $H(2,4)$ is the 4×4 rook's graph, one of the two strongly regular graphs with parameters $(16,6,2,2)$.

5. Named vertex-transitive graphs

Theorem 1 applies verbatim outside the Hamming family. Two cases gave sequences absent from the OEIS.

Petersen graph (spectrum $3^1, 1^5, (-2)^4$): $I = 3, 0, 6, 12, 36, 96, 264, 720, \dots$, with generating function $(1 - 2x + x^2 - 6x^3)/(1 - 2x - 2x^2)$ and $a(n) = 2a(n-1) + 2a(n-2)$ for $n > 3$, growth $(1 + \sqrt{3})^n$. Here $I(3) = 0$ because the girth is 5. Parent all-walk sequence: A091000. Submitted as A398309.

Icosahedron (spectrum $5^1, (\sqrt{5})^3, (-\sqrt{5})^3, (-1)^5$): $I = 5, 10, 40, 160, 740, 3360, 15540, 71560, \dots$, with generating function $(1 - 4x - 25x^4)/(1 - 4x - 5x^2 + 10x^3)$ and $a(n) = 4a(n-1) + 5a(n-2) - 10a(n-3)$ for $n > 4$. Although the eigenvalues are irrational, the walk counts are integral: $W(2m) = (25^m + 6 \cdot 5^m + 5)/12$ and $W(2m+1) = (5^{2m+1} - 5)/12$. Parent all-walk sequence: A054884. Submitted as A398310.

Two negative results keep the table honest: the octahedron $K_{2,2,2}$ gives $4, 8, 32, 96, 320, \dots$, which coincides with the catalogued A209084 and is therefore not new; and the complete and complete bipartite graphs give purely geometric rows (K_5 gives $4 \cdot 3^{L-2}$), which are degenerate.

6. The equivalential reading of the hypercube rows

The pure equivalential calculus is the propositional calculus whose only connective is the biconditional $\leftrightarrow$. A formula in k variables with L variable-occurrences is determined by two independent data: a binary tree shape with L leaves, and a word of length L over the k -letter alphabet labelling those leaves. It is a classical fact, used throughout the single-axiom literature on the equivalential calculus (e.g. Wos, Ulrich, and Fitelson [5]), that such a formula is a theorem exactly when every variable occurs an even number of times — a condition depending only on the word, not the tree. This is elementary over $GF(2)$: interpreting $a \leftrightarrow b$ as $1 + a + b$, a $\leftrightarrow$ -formula evaluates to $1 + \sum n_i x_i$ where n_i is the parity of the number of occurrences of variable i , so it is identically 1 iff every n_i is even.

Call a word of length L over k letters in which every letter occurs an even number of times a theorem-word. Reading each edge-step of Q_k as one of the k coordinate directions gives:

Proposition 4. The theorem-words of length L in k variables are exactly the closed walks of length L on Q_k based at a vertex. Consequently the first-return walks of length L on Q_k are exactly the irreducible theorem-words: theorem-words no nonempty proper prefix of which is already a theorem-word. The row $H(k,2)$ of the array therefore counts a graph-theoretic and a proof-theoretic object simultaneously.

(The count of theorems, as opposed to theorem-words, is the Catalan number $C(L-1)$ times the number of theorem-words, since the tree shape is free.)

This generalises along the family. For $H(k,q)$, a closed walk is a word over the alphabet of pairs (coordinate, nonzero increment) whose total content vanishes in $(\mathbb{Z}/q)^k$; these are the identities of the equational theory of $(\mathbb{Z}/q)^k$, and first returns are the irreducible such words. The $q = 2$ case is distinguished because there the group operation coincides with the biconditional.

7. Verification

All values were computed exactly in integer arithmetic and checked three independent ways: (i) the spectral formula (2); (ii) direct powers of the adjacency matrix, entry (v,v) ; and (iii) for the Hamming family, brute-force enumeration of words over the generator alphabet with vanishing content. The three agree in every case tested ($k \leq 3$, $q \leq 4$, $L \leq 8$). The equivalential dictionary of §6 was checked by enumerating all even-occurrence words directly for $k \leq 4$. Code accompanies this note.

8. Open

A closed form for the growth rate of the $H(k,q)$ rows as a function of k and q is not given here; it is the reciprocal of the smallest root of the numerator of (4), algebraic of degree equal to the number of distinct nonzero eigenvalues. For the hypercubes the all-walk growth rate is exactly k^2 , the first-return rate ρ_k is strictly smaller, and $\rho_k/k^2 \rightarrow 1$.

References

- [1] M. Bernstein and N. J. A. Sloane, Some canonical sequences of integers, *Linear Algebra Appl.* 226–228 (1995), 57–72; erratum 320 (2000), 210. Free: [arXiv:math/0205301](https://arxiv.org/abs/math/0205301) and oeis.org/A003633/a003633_1.pdf.
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- [3] C. Godsil and G. Royle, *Algebraic Graph Theory*, Springer, 2001. The single fact used (closed walks of length L equal $(1/N) \sum m_i \lambda_i^L$) is proved in §3.
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