A machine-checked proof of opacity for a Faber–Mycielski barrier

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Companion to A certified opaque barrier for the unit disc of length 4.79984937… (Zenodo, DOI 10.5281/zenodo.21568685). That note certifies the length of the barrier; this one machine-checks its opacity. The Lean source is lean/OpaqueDisc.lean.

Abstract

An opaque set for the unit disc is a set meeting every straight line that meets the disc. I report a formal verification, in Lean 4 with mathlib, that the three-piece Faber–Mycielski barrier is opaque. The main theorem barrier_isOpaque is proved with no sorry and depends only on the three standard axioms propext, Classical.choice, Quot.sound; in particular the axiom audit shows no sorryAx, so no gap remains anywhere in its dependency tree. The formalization is conditional in one explicit and deliberate respect: the two chord parameters enter as hypotheses (the feet lie on their chords), certified externally by interval arithmetic rather than inside the kernel. I am aware of no prior formalization of an opaque-set result — a search of the Archive of Formal Proofs topic listings, mathlib, and public Lean/Coq/Isabelle repositories found none — and I claim no new bound and no optimality here, only that a specific classical opacity argument now has a machine-checked proof. The reduction underlying it is formalized for an arbitrary subset of the plane, not merely the disc, so it applies unchanged to squares, polygons and any other body.

1. What is formalized

Everything is stated elementarily in $\mathbb{R} \times \mathbb{R}$ with the dot product written out, so the development depends on no inner-product-space packaging. A line with unit normal u θ and offset p, the closed unit disc, and opacity are

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def line (θ p : ℝ) : Set Pt := {x | dot x (u θ) = p}
def disc : Set Pt := {x | dot x x ≤ 1}
def IsOpaque (B : Set Pt) : Prop :=
  ∀ θ p, (line θ p ∩ disc).Nonempty → (line θ p ∩ B).Nonempty
```

The barrier is built from four gap angles t : Fin 4 → ℝ exactly as in the companion note (Lean's E t 0 … E t 3 are the note's apices $E_1 \dots E_4$): the covered arc, two bow tangent segments, and two altitude-foot arrows.

The main theorem:

```
theorem barrier_isOpaque
    (hcos : ∀ k, 0 < cos (t k / 2)) (hpi : ∀ k, t k / 2 ≤ π) (hgap : hGap t ≤ π)
    (hf1 : foot (E t 1) (E t 0) (E t 3) ∈ segment ℝ (E t 0) (E t 3))
    (hf2 : foot (E t 2) (E t 1) (E t 3) ∈ segment ℝ (E t 1) (E t 3)) :
    IsOpaque (barrier t)
```

with a corollary replacing the analytic hypotheses by rational bounds that norm_num discharges for the optimized parameters:

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theorem barrier_isOpaque_of_gap_bounds
    (hpos : ∀ k, 0 < t k) (hle : ∀ k, t k ≤ 5 / 4) (hf1 …) (hf2 …) :
    IsOpaque (barrier t)
```

2. Structure of the proof

Reduction (Lemma 3.1). opaque_iff_shadow_covers : B is opaque iff its shadow covers $[-1,1]$ in every direction. This replaces a two-parameter family of lines by a one-parameter family of directions, and it holds on the closed interval, so tangent lines need no separate limiting argument. The load-bearing step is line_meets_disc_iff — a line meets the disc iff $|p| \le 1$ — proved from the Lagrange identity $p^2 + (x_1 \sin\theta - x_2 \cos\theta)^2 = \|x\|^2$.

Apex reach (Lemma 3.3). apex_reach : the shadow reaches 1 in every direction. Two certified inequalities carry it: arc_reach (a point of the covered arc projects to exactly 1 on the arc's own range) and apex_reach_gap (within gap k, $E_k \cdot u(\theta) = \sec(t_k/2)\cos(m_k - \theta) \ge \sec(t_k/2)\cos(t_k/2) = 1$, by monotonicity of cosine). apex_reach_fund then splits any direction in the fundamental domain into the arc range or one of the four gaps — the gaps tile $[-h, h]$, which is phi4_eq_hGap — and apex_reach extends this to all of $\mathbb{R}$ by a 2π-periodic reduction via toIcoMod.

No gaps (Lemma 3.2). shadow_ordConnected : the shadow is order-connected. See §4.

The reduction needs nothing about the disc. Opacity for an arbitrary set K is formalized as IsOpaqueFor K B, and

```
theorem isOpaqueFor_iff_shadow_subset (K B : Set Pt) :
    IsOpaqueFor K B ↔ ∀ θ, shadow K θ ⊆ shadow B θ
```

holds with no convexity, compactness or boundedness hypothesis whatsoever; the disc statement is the instance K = disc, via shadow_disc : shadow disc θ = Icc (-1) 1. A companion monotonicity lemma,

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theorem IsOpaqueFor.mono (hsub : K' ⊆ K) (h : IsOpaqueFor K B) : IsOpaqueFor K' B
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says a barrier for K is a barrier for every subset of K — so the certified disc barrier is simultaneously an opaque set for every body inscribed in the unit disc, bounding all of their barrier lengths by 4.799849374678… at once. These are the parts of the development intended for reuse: any future certified bound for a square, a regular polygon or another convex body inherits a machine-checked validity criterion rather than needing its own.

Assembly (Theorem 3.4). Order-connectedness, plus a shadow point ≥ 1 from apex reach at θ and one ≤ -1 from apex reach at $\theta + \pi$ (where dot_u_add_pi negates every projection), fills $[-1,1]$. The reduction finishes.

3. The formalization boundary

I state plainly what is and is not inside the kernel.

Inside. The reduction, the apex identity $\|E_k\| = \sec(t_k/2)$, the reach inequalities, the gap tiling, the periodic reduction, order-connectedness of the shadow, and the final assembly. Also the passage from rational gap bounds to the analytic hypotheses (barrier_isOpaque_of_gap_bounds).

Outside. The two feet-on-chord conditions hf1, hf2 appear as hypotheses. They are the formal counterpart of the certified scalars $s_1 = 0.294660041656711\ldots$, $s_2 = 0.467335706033416\ldots$, each enclosed strictly in (0,1) by outward-rounded interval arithmetic in verify-upper/opacity_shadow_certified.py. Discharging them inside Lean would require certified evaluation of cosine and sine at specific transcendental arguments in the kernel; that is a separate undertaking and I have not done it. So the Lean theorem is a proof of the implication, and the two remaining numerical facts rest on the external certified computation — with margins of order 10^{-1}, not marginal.

Also outside. The length 4.799849374678549857544655408276… is not formalized. It is an interval enclosure from the companion note and plays no part in the opacity proof.

4. A note on Lemma 3.2

The reformulation of this lemma is the one methodological remark worth recording.

Stated as in the companion note — the shadow is a single compact interval [m(θ), M(θ)] — the lemma invites a proof by compactness and connectedness of the five pieces, and in a proof assistant that is genuinely heavy. It is also more than the theorem needs.

The certified computation does not prove that statement. It treats the barrier as three objects: a connected piece (the arc together with the two bows, glued at shared endpoints, so its projection is a single interval with no gap at the gluings) and the two arrows. The zero-margin handoffs are closed by the foot-on-chord identity

```
(F − E_i)·u  ·  (F − E_j)·u  =  −s(1−s) ((E_j − E_i)·u)²  ≤  0,
```

i.e. the foot lies on its chord, so its projection is pinned between the projections of two apices already lying in the connected piece. The overlap is forced by the construction; there is no gap to close and no topological obstruction to clear.

Restating Lemma 3.2 as OrdConnected (shadow (barrier t) θ) — which is exactly what the assembly consumes, and which drops a boundedness claim never used — makes the Lean proof follow that same structure and collapse to elementary parts:

- dot_mem_segment_proj — the projection is linear, so a point of a segment projects between the projections of its endpoints. This single fact is the mechanism.
- shadow_segment — a segment's shadow is the interval between endpoint projections.
- shadow_arc_ordConnected — the arc's shadow is a continuous image of an interval. This is the only continuity step in the development.
- ordConnected_union_of_mem — two order-connected sets of reals sharing a point union to an order-connected set.
- foot_proj_mem_of_ordConnected — a foot on its chord, with both chord endpoints projecting into an order-connected set, projects into that set. The arrow cannot miss.

The gluing then runs in the order the certified computation runs it: bows to the arc at shared endpoints, then arrow_1 via its foot on the chord $[E_0, E_3]$, then arrow_2 via its foot on $[E_1, E_3]$ (whose endpoint E_1 is already attached inside arrow_1).

The lesson generalizes past this problem: when a numerical certificate and a formal proof disagree about which step is hard, the certificate is usually right, and the formal statement is usually asking for more than the theorem needs.

5. Reproducibility

Toolchain Lean 4.32.1 with mathlib v4.32.1. Source lean/OpaqueDisc.lean; lake build completes with no sorry warning. The axiom audit is

```
#print axioms OpaqueDisc.barrier_isOpaque
'OpaqueDisc.barrier_isOpaque' depends on axioms: [propext, Classical.choice, Quot.sound]
```

The external certificates are verify-upper/opacity_shadow_certified.py (shadow bounds, apex identity, foot-on-chord scalars), verify-upper/beam_3arc.py (length), and an independent per-line test over 9,003,000 lines in hunt/opacity.py.

References

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