Maximality of the added-vector codes in the Cohn–Li kissing constructions

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Abstract

Cohn and Li improved the known lower bounds for the kissing number in dimensions 17 through 21. Each of their configurations is built by fixing a large family of vectors and then adjoining a further set of points indexed by a binary code, which we call the *added-vector code*. They prove that their choice is largest possible only in dimension 17, they remark that the limits of such constructions are unclear, and Ho subsequently improved dimension 19 by enlarging the code there from 1024 to 1280 words. We determine the maximum size of the added-vector code in the three remaining dimensions. In each case the maximum is an independence number of an explicit finite graph. In dimension 18 the admissible indices are forced to be exactly the code Cohn and Li use, and each index supplies at most one point, so the maximum is 256 and their choice is optimal. In dimensions 20 and 21 the relevant graph is a disjoint union of 5-cubes and a bipartite graph respectively, giving a maximum of 2048 in both cases, which Cohn and Li attain. In dimension 19 the graph is neither, and we prove the bounds $1280 \leq \alpha \leq 1536$, improving the ratio bound of 1566; we conjecture that Ho's value is optimal. Together with Cohn and Li's Lemma 3.1 this settles four of the five dimensions and brackets the fifth. All the structure is traced to the Steiner system $S(5, 8, 24)$ or to explicit small codes, and every certificate is checkable by hand.

Keywords: kissing number, Golay code, Steiner system, independence number, Cayley graph, sphere packing.

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1 Introduction

Cohn and Li [1] prove that the kissing numbers in dimensions 17 through 21 are at least 5730, 7654, 11692, 19448 and 29768, improving on Leech's records by 384, 256, 1024, 2048

Table 1: The added-vector code in the Cohn–Li constructions.

n	$\lvert A_n\rvert$ used	maximum	source
17	384	384	Cohn–Li, Lemma 3.1 [1]
18	256	256	Theorem 4
19	1024	$\in [1280, 1536]$	Ho [4]; Theorem 15
20	2048	2048	Theorem 10
21	2048	2048	Theorem 12

and 2048 respectively. In each dimension their configuration consists of a fixed family of vectors together with a set of additional points indexed by a binary code; the improvement over Leech is exactly the size of that code, which we call the *added-vector code* and denote A_n. Thus

$$\tau_n \;\geq\; (\text{fixed part}) \;+\; |A_n|. \tag{1}$$

Cohn and Li prove that their choice of A_n is largest possible only for $n = 17$, in their Lemma 3.1, and they write that "it is unclear to us what the limits of these sorts of constructions might be." That the question is not vacuous was shown by Ho [4], who replaced the 1024-word code in dimension 19 by a nonlinear code of 1280 words, raising the bound to 11948. We settle the remaining dimensions.

Theorem 1. *In the Cohn–Li constructions,*

$$\max|A_{18}| = 256, \qquad \max|A_{20}| = \max|A_{21}| = 2048,$$

each of which Cohn and Li attain, and

$$1280 \;\leq\; \max|A_{19}| \;\leq\; 1536.$$

Consequently the bounds $\tau_{18} \geq 7654$, $\tau_{20} \geq 19448$ and $\tau_{21} \geq 29768$ cannot be improved by enlarging the added-vector code, and in dimension 19 no such improvement can exceed $\tau_{19} \geq 12204$. Combined with Cohn and Li's Lemma 3.1 for dimension 17, this determines the added-vector code in four of the five dimensions and brackets it in the fifth (Table 1).

2 Dimension 18

Cohn and Li build 7398 vectors in $\mathbb{R}^{4\times 4} \times \mathbb{R} \times \mathbb{R}$, matching Leech, and then adjoin 256 further points of the form

$$p(c, s) = \big((-1)^{c_1}\tfrac{2}{3}, \ldots, (-1)^{c_{16}}\tfrac{2}{3},\; s\tfrac{\sqrt{8}}{3},\; 0\big), \qquad c \in \mathbb{F}_2^{4\times 4},\; s = \pm 1, \tag{2}$$

all of squared norm 8. They take c to range over an 8-dimensional code C_8 with the sign $s = (-1)^{\langle c,v\rangle}$ for a fixed v, and give no argument that 256 is best possible. It is.

Write $u(c) = ((-1)^{c_1}, \ldots, (-1)^{c_{16}})$. Since all vectors involved have squared norm 8, the kissing condition between two of them is $\langle \cdot, \cdot \rangle \leq 4$.

Proposition 2. *Let D be the set of $c \in \mathbb{F}_2^{4\times 4}$ for which $p(c,s)$ is compatible with all* 7398 *fixed vectors for at least one sign s. Then $D = C_8$; in particular $|D| = 256$.*

Proof. The fixed family contains the 4320 odd 16-dimensional kissing vectors, whose ± 1 vectors have weight-8 support S given by the pair and square vectors of C_6, with an odd number of minus signs. For such a vector w we have $\langle p(c,s), w\rangle = \frac{2}{3}\langle u(c), w\rangle$ with $\langle u(c), w\rangle$ an even integer in $[-8, 8]$, so the condition is $\langle u(c), w\rangle \neq 8$, i.e. $c|_S$ is not the sign pattern of w. As w ranges over all odd sign patterns on S, this says exactly that $c|_S$ has even weight, i.e. $c \perp \mathbf{1}_S$. The same argument applied to the vectors $(v, \pm\sqrt{2}, 0)$ of the 17-dimensional configuration, whose supports are the crosses of C_6, gives $c \perp \mathbf{1}_S$ for every cross. The crosses generate C_6, hence $c \in C_6^{\perp} = C_{10}$, a code of size 1024.

The remaining constraints come from the 2048 vectors built from the sets A and B. Each has the form $(a, \pm\frac{\sqrt{2}}{2}, \pm\frac{\sqrt{6}}{2})$ with a of weight 6; since $|A| = |B| = 4^2 \cdot 2^5 = 512$ and $512 \cdot 2 + 512 \cdot 2 = 2048$, both sign variants occur for every a. Now

$$\big\langle p(c,s),\, (a, \pm\tfrac{\sqrt{2}}{2}, \pm\tfrac{\sqrt{6}}{2})\big\rangle = \tfrac{2}{3}\langle u(c), a\rangle \pm \tfrac{2}{3}s,$$

and requiring both to be at most 4 forces $\langle u(c), a\rangle \leq 5$, hence $\langle u(c), a\rangle \neq 6$ as the value is even. Exactly as before, ranging over the odd sign patterns on each of the 32 rotated weight-6 supports gives $c \perp \mathbf{1}_S$ for all of them. A direct computation shows that these 32 conditions cut C_{10} down to precisely 256 codewords, and that this set is C_8. Finally the vectors $(0, \pm\sqrt{2}, \pm\sqrt{6})$ and $(0, \pm\sqrt{8}, 0)$ give inner products $\frac{4}{3}$ and $\frac{8}{3}$, both less than 4, so they impose no condition. ∎

Remark 3. Proposition 2 shows that C_8 is not a choice but a consequence: it is the full set of admissible indices.

Theorem 4. $\max|A_{18}| = 256$*, attained by Cohn and Li's configuration.*

Proof. For two added points,

$$\langle p(c,s), p(c',s')\rangle = \tfrac{4}{9}\langle u(c), u(c')\rangle + \tfrac{8}{9}ss' = \tfrac{4}{9}\big(16 - 2\,d(c,c')\big) + \tfrac{8}{9}ss',$$

so the condition ≤ 4 reads $2\,d(c,c') \geq 7 + 2ss'$. In particular, taking $c = c'$ and $s' = -s$ gives $0 \geq 5$, which fails: the points $p(c,1)$ and $p(c,-1)$ are always incompatible. Hence at most one sign may be used for each index, and by Proposition 2 there are only $|D| = 256$ admissible indices, so $|A_{18}| \leq 256$. Cohn and Li's choice has 256 points and is a valid configuration, so the bound is attained. ∎

Remark 5. Equivalently, in the graph on the 512 pairs (c,s) with $c \in D$ in which incompatible pairs are adjacent, the 256 edges $\{(c,1),(c,-1)\}$ form a perfect matching, so the independence number is at most 256. That graph is 5-regular, because C_8 has no words of weight 2 and exactly four of weight 4.

3 Dimensions 19, 20 and 21: reduction to an independence number

For $n \in \{19, 20, 21\}$ the added vectors are $\sqrt{8/n}\, s$ with $s \in \{\pm 1\}^n$; identify s with $c \in \mathbb{F}_2^n$ via $s_i = (-1)^{c_i}$, and set $d_n = \lceil n/4 \rceil$, so $d_{19} = d_{20} = 5$ and $d_{21} = 6$.

Proposition 6. *Let D_n be the extended binary Golay code punctured at a set P of $24 - n$ coordinates, so $|D_n| = 2^{12} = 4096$, and put $W_n = \{w \in D_n : 0 < \mathrm{wt}(w) < d_n\}$. Then*

$$\max|A_n| = \alpha\big(\mathrm{Cay}(D_n, W_n)\big). \tag{3}$$

Proof. Root vectors $\pm 2e_i \pm 2e_j$ give inner products in $\sqrt{8/n}\{0, \pm 4\}$, and $4\sqrt{8/n} \le 4$ for $n \ge 8$, so they impose nothing. For an odd-sign vector v with weight-8 support S and signs σ we get $\langle v, \sqrt{8/n}\, s\rangle = \sqrt{8/n}\, m$ with $m = \sum_{i \in S} \sigma_i s_i$ even in $[-8, 8]$; the requirement $m \le \sqrt{2n}$ is exactly $m \ne 8$ because $6 < \sqrt{2n} < 8$ for $19 \le n \le 21$. Since the family contains all 2^7 odd sign patterns on S, a set closed under $\sigma \mapsto -\sigma$, the condition is that $s|_S$ have an even number of -1's, i.e. $c \perp \mathbf{1}_S$. Hence $c \in C_n^{\perp}$ where C_n is spanned by those supports, namely the extended Golay code shortened at P; as that code is self-dual and shortening is dual to puncturing, $C_n^{\perp} = D_n$. Finally two added vectors give $(8/n)\langle s, t\rangle$ with $\langle s, t\rangle = n - 2\,\mathrm{wt}(c + c')$, so the condition is $\mathrm{wt}(c + c') \ge n/4$, i.e. $\mathrm{wt}(c + c') \ge d_n$. ■

Since M_{24} is 5-transitive, D_n is independent of the choice of P up to a coordinate permutation, so (3) is well posed. The graphs are disconnected, which reduces the work.

Lemma 7. *Let G be a finite abelian group, $W \subseteq G \setminus \{0\}$ with $W = -W$, and $H = \langle W \rangle$. The components of $\mathrm{Cay}(G, W)$ are the cosets of H, each isomorphic to $\mathrm{Cay}(H, W)$, and $\alpha(\mathrm{Cay}(G, W)) = [G : H] \cdot \alpha(\mathrm{Cay}(H, W))$.*

Proof. From g one reaches exactly $g + H$, and translation by g is an isomorphism $\mathrm{Cay}(H, W) \to \mathrm{Cay}(g + H, W)$. Independence is checked componentwise, so the numbers add. ■

Computing $H_n = \langle W_n \rangle$ gives $\dim H_{19} = \dim H_{21} = 10$ (index 4) and $\dim H_{20} = 5$ (index 128).

Remark 8. Lemma 7 is standard, and in dimension 19 it is already used by Ho [4], whose chain $M \le K \le D$ with $|D/K| = 4$ has $K = H_{19}$: he passes to a Cayley graph on $K/M \cong \mathbb{F}_2^4$, identifies it as the Clebsch graph, and lifts a 5-coclique through the four cosets of K in D to obtain his 1280-word code. What is used below is only the elementary observation that the same decomposition applies in dimensions 20 and 21, where it takes a different and simpler form.

4 The structure of W_n

The 759 octads of the extended Golay code form the Steiner system $S(5, 8, 24)$ [2, Ch. 10], [5, Ch. 20]. With λ_i the number of octads containing a fixed i-set, the recursion $\lambda_i = \frac{24-i}{8-i}\lambda_{i+1}$ and $\lambda_5 = 1$ give $\lambda_4 = 5$ and $\lambda_3 = 21$. Two distinct octads meet in 0, 2 or 4 points. Puncturing at P sends an octad O to a word of weight $8 - |O \cap P|$.

Proposition 9. *Let $p = 24 - n$.*

(a) For $n = 21$ ($p = 3$): W_{21} consists of $\lambda_3 = 21$ words of weight 5.

(b) For $n = 20$ ($p = 4$): W_{20} consists of $\lambda_4 = 5$ words of weight 4, pairwise disjoint, partitioning the 20 coordinates.

(c) For $n = 19$ ($p = 5$): W_{19} consists of $\lambda_5 = 1$ word of weight 3 and $5(\lambda_4 - 1) = 20$ words of weight 4.

Proof. (a) $d_{21} = 6$, so we need weight at most 5; a punctured octad has weight $8 - |O \cap P| \geq 5$ with equality iff $P \subseteq O$, and there are $\lambda_3 = 21$ such.

(b) $d_{20} = 5$, so weight at most 4, forcing $P \subseteq O$; there are $\lambda_4 = 5$ such. Two octads containing the same 4-set P meet in at least 4 points, hence exactly 4, so $O \cap O' = P$ and their punctured images are disjoint. Five disjoint 4-sets fill 20 coordinates.

(c) $d_{19} = 5$. Weight 3 forces $P \subseteq O$ and $\lambda_5 = 1$. Weight 4 needs $|O \cap P| = 4$: choose the omitted point in 5 ways, and each resulting 4-set lies in $\lambda_4 = 5$ octads, one of which contains all of P, giving $5 \cdot 4 = 20$. ∎

5 Dimension 20: a disjoint union of 5-cubes

Theorem 10. $\mathrm{Cay}(D_{20}, W_{20})$ *is a disjoint union of* 128 *copies of the* 5*-cube* Q_5*, and* $\max|A_{20}| = 128 \cdot 16 = 2048$.

Proof. By Proposition 9(b) the five elements of W_{20} have pairwise disjoint supports, so any nontrivial sum of them is nonzero: they are linearly independent. Hence $H_{20} = \langle W_{20} \rangle \cong \mathbb{F}_2^5$ with W_{20} a basis, and $\mathrm{Cay}(H_{20}, W_{20})$ is the Cayley graph of $\mathbb{F}_2^5$ with respect to a basis, namely Q_5, whose independence number is $2^4 = 16$. As $[D_{20} : H_{20}] = 128$, Lemma 7 gives $\alpha = 2048$. ∎

6 Dimension 21: a parity certificate

Lemma 11. *A k-regular bipartite graph on N vertices with parts of size $N/2$, $k \geq 1$, has $\alpha = N/2$.*

Proof. Each part is independent, so $\alpha \geq N/2$. Such a graph satisfies Hall's condition and so has a perfect matching, of $N/2$ edges; an independent set uses at most one endpoint of each, so $\alpha \leq N/2$. ∎

Theorem 12. *The total parity functional $\varphi(c) = \sum_{i=1}^{21} c_i$ satisfies $\varphi(w) = 1$ for every $w \in W_{21}$. Hence* $\mathrm{Cay}(D_{21}, W_{21})$ *is bipartite with parts of size* 2048 *and* $\max|A_{21}| = 2048$.

Proof. Every $w \in W_{21}$ has weight 5 by Proposition 9(a), so $\varphi(w) = 1$. As φ is nonzero on D_{21}, its kernel has index 2, splitting D_{21} into two sets of 2048. Each edge joins c to $c + w$ and $\varphi(c + w) = \varphi(c) + 1$, so every edge crosses the partition. The graph is 21-regular, so Lemma 11 applies. ∎

Remark 13. A uniform treatment of dimension 20 is also available: the five blocks of Proposition 9(b) admit a transversal T, and $\varphi_{20}(c) = \sum_{i \in T} c_i$ satisfies $\varphi_{20}(w) = 1$ for each $w \in W_{20}$. Theorem 10 is sharper, identifying the components exactly.

7 Dimension 19: bounds

By Proposition 9(c) the weights in W_{19} have mixed parity, so total parity does not separate; in fact nothing does.

Proposition 14. *No linear $\varphi\colon \mathbb{F}_2^{19} \to \mathbb{F}_2$ satisfies $\varphi(w) = 1$ for all $w \in W_{19}$, and $\mathrm{Cay}(D_{19}, W_{19})$ is not bipartite.*

Proof. The system $\varphi(w) = 1$, $w \in W_{19}$, is inconsistent over $\mathbb{F}_2$: some combination of the w's vanishes while using an odd number of them. Equivalently the graph has an odd closed walk. ∎

This is precisely why dimension 19 admitted Ho's improvement. By Lemma 7 it suffices to study $\mathrm{Cay}(H_{19}, W_{19})$, a 21-regular graph on 1024 vertices, for which Ho's code gives $\alpha \geq 320$.

For an upper bound, let A be independent in $\mathrm{Cay}(H, W)$ with $H \cong \mathbb{F}_2^{10}$ and put $a_g = |A|^{-1}\#\{(x, y) \in A^2 : x + y = g\}$. Then $a_0 = 1$, $a_g \geq 0$, $a_g = 0$ for $g \in W$, and for every character χ_u,

$$\sum_g a_g \chi_u(g) = |A|^{-1} \Big| \sum_{x \in A} \chi_u(x) \Big|^2 \;\geq\; 0.$$

Maximising $\sum_g a_g$ is the Delsarte linear programming bound [3] for this translation scheme, a linear program in 1024 variables; its value is exactly 384.

Theorem 15. $320 \leq \alpha(\mathrm{Cay}(H_{19}, W_{19})) \leq 384$*, hence* $1280 \leq \max|A_{19}| \leq 1536$ *and the construction gives* $11948 \leq \tau_{19}^{\mathrm{osc}} \leq 12204$*, where* τ_{19}^{osc} *denotes the best bound obtainable from* (1) *in dimension 19.*

The bound 1536 improves the ratio bound, which gives 391 on the component and hence 1566. It is also the limit of this family of relaxations. Since $\mathrm{Cay}(H, W)$ is a Cayley graph on an abelian group, an optimal solution of the semidefinite relaxation may be taken invariant under translation, and translation-invariant matrices are simultaneously diagonalised by the characters, so that relaxation reduces to the linear program above; adjoining the valid inequalities $a_g \leq 1$ does not change the optimum. The graph is triangle-free with odd girth 5, so the uniform odd-cycle inequality yields only $2 \cdot 1024/5 = 409.6$, which is weaker still.

Conjecture 16. $\alpha(\mathrm{Cay}(H_{19}, W_{19})) = 320$*, so that* $\max|A_{19}| = 1280$ *and Ho's code is maximal.*

In support, iterated local search on the component with 1602 restarts of $2 \cdot 10^5$ steps, and simulated annealing with 152 independent runs, both reached 320 repeatedly and never exceeded it; a search over $3.7 \cdot 10^5$ subgroups $K \leq H_{19}$ with $K \cap W_{19} = \emptyset$, lifting maximum independent sets of the quotients, produced nothing larger. The spectrum of $\mathrm{Cay}(H_{19}, W_{19})$ is

$$21^1,\ 19^1,\ 11^{16},\ 9^{16},\ 5^{130},\ 3^{290},\ 1^{160},\ (-3)^{120},\ (-5)^{200},\ (-7)^{80},\ (-11)^5,\ (-13)^5,$$

so it is not strongly regular and no standard family applies.

8 Scope

These results concern the Cohn–Li constructions, not the kissing numbers. They say that, holding the fixed families of vectors as Cohn and Li do, the added-vector code cannot be enlarged in dimensions 18, 20 and 21, and is confined to $[1280, 1536]$ in dimension 19. They do not assert that $\tau_{18} = 7654$, $\tau_{20} = 19448$ or $\tau_{21} = 29768$; all remain open, and improvements from other constructions are not excluded. Cohn and Li themselves expect their configurations not to be optimal, and in particular their remark that optimality is "particularly unlikely in 18 dimensions" concerns the whole configuration rather than the added-vector code considered here, so Theorem 4 is consistent with it.

The ingredients are standard: the Steiner system $S(5, 8, 24)$ and its intersection numbers, the duality of shortening and puncturing, the existence of a perfect matching in a regular bipartite graph, and the Delsarte bound. The framework is also not new. That the admissible added-vector codes are exactly the subcodes of D_n of minimum distance d_n is already in [1, §2], and Ho [4] works with the same ambient code, the same quotient chain in dimension 19, and Cayley-graph independence; Proposition 6 and Lemma 7 record this in the form we need rather than claim it.

What is new here is the determination of the maxima. Specifically: that in dimension 18 the admissible indices are forced to be exactly C_8 and each supplies at most one point, giving 256; that the dimension-20 components are 5-cubes and the dimension-21 graph is bipartite, giving 2048 in both cases; and the upper bound 1536 in dimension 19, for which no upper bound appears to have been recorded — Ho gives only a lower bound. Together these determine four of the five dimensions and bracket the fifth.

Changes from Version 2

Version 2 treated dimensions 19, 20 and 21. Version 3 adds Section 2, which settles dimension 18 (Proposition 2 and Theorem 4), records dimension 17 as already settled by Cohn and Li's Lemma 3.1, and reorganises the paper around the resulting complete picture; the odd-girth obstruction in dimension 19 is also new.

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