Rigorous areas for the classical Lebesgue universal covering ladder

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Abstract

Lebesgue's universal covering problem, posed in 1914, asks for the convex set of least area containing an isometric copy of every planar set of diameter 1. The known upper bounds descend through a ladder of constructions — the regular hexagon, Pál (1920), Sprague (1936), Hansen (1992), Baez–Bagdasaryan–Gibbs (2015), Gibbs (2018) — whose areas are quoted in the literature to at most twelve decimal places. So far as I can determine none had been evaluated with rigorous error control. This note certifies the whole ladder:

hexagon	√3/2	
Pál	2 − 2/√3	
Sprague	0.844137708435197570894066994044…	
Hansen	0.844137708397690336757182874586…	
BBG 2015	0.844115376859376746806104420762…	(σ = 1.3)
0.844115297128419059214192192376…	(Egan's σ)	
Gibbs 2018	a ≤ 0.8440935944, certified with margin 9.3×10^{-12}	

The last line is, to my knowledge, the first verification of the record since its publication eight years ago; together with the certified lower bound of Xie (2026), the problem's bracket 0.833 ≤ a ≤ 0.8440935944 now rests on certified computation at both ends. Section 3 reconstructs Sprague's reduction and records a point of attribution: his 1936 paper proves only that Pál's cover is not minimal, and states no area. Section 4 describes the arithmetic. Section 5 confirms every digit of the Baez–Bagdasaryan–Gibbs table and settles a discrepancy with a figure circulated on the Azimuth blog. Sections 6 and 7 certify the 2015 cover and the 2018 record, the latter by running the record's five-step boundary map in second-order jet arithmetic over interval coefficients to bound the discretisation error rigorously. Section 8 reports a soundness defect in the kernel's first conversions — caught by a consistency gate between independently certified rungs — with its fix and its blast radius. Universality, that each set does cover every diameter-1 set, is taken from the literature throughout; the computational side conditions of the 2015 construction are certified here.

1. The ladder

A universal covering is a convex planar set containing an isometric copy of every set of diameter 1. Write a for the infimum of their areas. The regular hexagon with inscribed circle of diameter 1 is universal, so a ≤ √3/2. Pál removed two corners cut off by an inscribed dodecagon, giving a ≤ 2 − 2/√3. Sprague removed a further region near a third corner; Hansen removed two microscopic slivers; Baez–Bagdasaryan–Gibbs and then Gibbs improved the construction again. Against these, Brass and Sharifi proved a ≥ 0.832, certified and improved to a ≥ 0.833 by Xie [10] in June 2026, so that with the results of this note

$$0.833 \le a \le 0.8440935944,$$

and both ends of the bracket now rest on certified computation.

Every upper bound in that chain is a number rather than a closed form, and the arithmetic producing them is more delicate than it looks. Hansen's own published value for his second sliver was too large by three orders of magnitude — an error undetected for twenty-three years, and one whose mechanism is visible in §5. That is the motivation for evaluating the ladder with rigorous error control rather than in floating point.

2. Notation

Place the hexagon with centre at the origin and a vertex A_1 at the top. Throughout, $s = \sqrt{3}$ and $c = (s - 1)/2$, and the hexagon's vertices are

$$A_1 = (0, 1/s) \quad B_1 = (1/2, 1/(2s)) \quad C_1 = (1/2, -1/(2s))$$
$$D_1 = (0, -1/s) \quad E_1 = (-1/2, -1/(2s)) \quad F_1 = (-1/2, 1/(2s)).$$

Pál's cover replaces the corners at C_1 and E_1 by chords of the inscribed dodecagon. The two cut points that matter below are

$$C_2 = (c, -c) \quad \text{and} \quad E_3 = (-c, -c).$$

3. Sprague's reduction

3.1 What Sprague proved, and what he did not

A point of attribution first, because it affects how the rung should be described. Sprague's paper is three short sections. §1 gives a fresh proof that the regular hexagon of side $1/\sqrt{3}$ is a covering — he calls such a set a Tafel; §2 reproduces Pál's reduction to the octagonal cover T of area $2 - \tfrac{2}{3}\sqrt{3} \approx 0{,}8453$, obtained by cutting two non-adjacent corner triangles ("es gibt immer zwei, die nicht benachbart sind"); and §3, headed *Die Tafel T ist keine Tafel kleinsten Flächeninhaltes*, shows that a further piece may be cut from a third corner — the part lying outside the circle of radius 1 about a cut vertex — and concludes "Also ist T′ eine Tafel und darum T keine Minimaltafel."

Sprague states no area for T′. His result is qualitative: Pál's cover is not minimal. The figure 0.844137708436, attributed to Sprague throughout the modern literature, is therefore not his; it is a later evaluation of his construction. What follows should be read as an evaluation of Sprague's construction, not as a check of a number he published.

His combinatorics is exactly the one used below: two non-adjacent corner triangles removed, and the third non-adjacent corner reduced by unit circles centred at cut vertices. In the labelling of §2 those are the corners C_1, E_1 and A_1.

3.2 The construction

Baez–Bagdasaryan–Gibbs describe the step as follows: near A_1, the region lying outside the unit circle centred at E_3 may be removed, as may the region lying outside the unit circle centred at C_2. Sprague's own §3 describes cutting at a single circle and then invokes the symmetry of the configuration; whether his cover is understood to have one piece removed or two is not determinable from his text without his figure, and the two-piece reading is the one that reproduces the attributed value.

The qualifier near A_1 carries the whole content, and it is where a reconstruction can go wrong. The set of points of Pál's cover lying outside the circle centred at E_3 is not connected: it has one component adjacent to A_1 and further components elsewhere in the cover. Only the component at A_1 is removed. Deleting the whole exterior would destroy universality, and deleting only the points outside both circles would delete far too little. So the removed set is

```
cap = (component at A₁ of the exterior of circle(E₃,1))
∪ (component at A₁ of the exterior of circle(C₂,1)),
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a union of two overlapping regions, illustrated in Figure 1.

3.3 Why this reading and not another

Three readings of the prose are available, and they are discrete alternatives rather than settings of a tunable parameter — which matters, because matching a published number by adjusting a continuous knob would be worthless as evidence. Measuring each against the area the attributed figure requires (Pál minus 0.844137708436, or 0.001161753185):

reading	area	ratio to required
outside both circles (intersection of exteriors)	0.000168984	0.145
outside either circle, over the whole cover	0.034192698	29.4
union of the two components at A_1	0.001159759	0.9983

The first two are not near misses; they are wrong by factors of about seven and thirty. The third agrees to the accuracy of the grid used to measure it, and when the same region is computed in closed form (§3.5) the agreement improves to $8\cdot10^{-13}$. No parameter was adjusted to achieve this: the region is fixed by the construction, and its area is a consequence.

3.4 Three checks independent of the area

Agreement with a published number is strong evidence but is not, by itself, proof that the reconstruction is the intended one. Three further checks bear on the geometry directly and do not involve the area at all.

(i) An exact tangency the reconstruction predicts.

> Lemma 1. *The unit circle centred at E_3 is tangent to the edge A_1B_1, meeting it at the single point $T = ((2 - s)/2, 1/2)$.*
>
> Proof. Edge A_1B_1 lies on the line $x + s\cdot y = 1$, whose unit normal is $(1, s)/2$. The signed distance from $E_3 = (-c, -c)$ to that line is
>
> $(-c - s\cdot c - 1)/2 = (-c(1 + s) - 1)/2 = (-1 - 1)/2 = -1,$
>
> because $c(1 + s) = (s - 1)(s + 1)/2 = 1$. The distance is therefore exactly 1, so the circle meets the line in one point, the foot of the perpendicular, namely $E_3 + (1, s)/2 = ((2 - s)/2, 1/2)$.

Nothing was fitted here. The circle's radius is 1 because a set of diameter 1 containing E_3 lies within distance 1 of it; the edge is a side of the hexagon; the tangency is a consequence, and it holds identically rather than approximately. A misidentified centre would not produce a distance of exactly 1. By reflection in the vertical axis the circle centred at C_2 is tangent to edge A_1F_1 at $T' = (-(2 - s)/2, 1/2)$.

(ii) The predicted vertex. Baez–Bagdasaryan–Gibbs state that after this reduction the cover has a vertex at the point X where the two arcs meet. The reconstruction produces exactly such a vertex, at

$$X = (0, -c + \sqrt{1 - c^2}) = (0, 0.564579455\ldots),$$

and both incidence conditions $|X - E_3| = |X - C_2| = 1$ hold to the precision of the arithmetic (§4). This is a structural prediction of the source text, confirmed without reference to any area.

(iii) Degeneration from the 2015 construction. The Baez–Bagdasaryan–Gibbs cover is built from a hexagon rotated by 30° + σ, with the analogous removal made at two points O and N depending on σ. As $\sigma \to 0$ one has $O \to C_2$ and $N \to E_3$, and their reduction becomes Sprague's. The reconstruction here is the σ = 0 member of that family, which is an independent route to the same region.

3.5 Closed form

Every boundary point is a surd, and I give them explicitly rather than by reference. The circle centred at E_3 meets edge A_1F_1, which lies on the line $-x + s\cdot y = 1$, where

$$4x^2 + (2s + 2)x + (7 - 4s) = 0,$$

obtained by substituting $y = (x + 1)/s$ into $(x + c)^2 + (y + c)^2 = 1$ and clearing denominators. The relevant root — the one near A_1, the other lying far outside the hexagon — is

$$x_P = [-(2 + 2s) + \sqrt{72s - 96}] / 8 = -0.013268602616\ldots,$$

giving $P = (x_P, (x_P + 1)/s)$, and by reflection $Q = (-x_P, (x_P + 1)/s)$ on edge A_1B_1. The two circles meet on the axis of symmetry at X as above. So the six points entering the computation — A_1, T, T′, P, Q, X — are all expressible in the field $\mathbb{Q}(\sqrt{3}, \sqrt{72\sqrt{3}-96})$, and the only transcendental quantities appearing anywhere are the arcsines used for circular segments.

Reading Figure 1(b), the component associated with E_3 is bounded by the edge A_1F_1 from P up to A_1, the edge A_1B_1 from A_1 down to the tangency T, and the arc of circle(E_3,1) from T back to P. Because both P and T lie on that circle while A_1 lies outside it, the arc bulges towards A_1, cutting the triangle P A_1 T into a part inside the circle and a part outside; the component is the outside part. Hence

$$\text{single} = \text{area}(\triangle\, P\, A_1\, T) - \text{seg}(P, T),$$

where seg(U, V) denotes the area between the chord UV and its minor arc on a unit circle,

$$\text{seg}(U, V) = (\theta - \sin\theta)/2, \quad \theta = 2\cdot\arcsin(|UV| / 2).$$

The two components overlap in the kite A_1 Q X P — bounded above by the two hexagon edges meeting at A_1 and below by the two arcs meeting at X — from which the two small segments on chords QX and XP must again be subtracted:

$$\text{overlap} = \text{area}(\triangle\, A_1\, Q\, X) + \text{area}(\triangle\, A_1\, X\, P) - \text{seg}(Q, X) - \text{seg}(X, P).$$

By inclusion–exclusion, and using the mirror symmetry that makes the two components congruent,

```
cap = 2 · single − overlap.
```

Numerically single = 0.000665362468…, overlap = 0.000168971750…, so cap = 0.001161753185…, and Pál's area less this cap agrees with Sprague's published 0.844137708436 to twelve significant figures. Section 6 gives the enclosure.

4. The arithmetic

Rather than assert that the computation avoids floating point, I state exactly what is represented and what is trusted.

Representation. Every quantity on the proof path is an interval whose two endpoints are exact rationals (Python Fractions, arbitrary precision). Arithmetic is interval arithmetic with outward rounding, so each operation returns a superset of the true result: an enclosure is never narrowed by an approximation.

Square roots, by integers alone. For a rational $q = a/b > 0$ write $\sqrt{a/b} = \sqrt{ab}/b$. With N a fixed power of ten and $m = \lfloor\sqrt{ab \cdot N^2}\rfloor$ computed by integer square root, the definition of the integer square root gives $m \le \sqrt{ab \cdot N^2} < m + 1$, hence

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m/(bN) ≤ √(a/b) ≤ (m + 1)/(bN).
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The enclosure is therefore provable by construction and uses no arithmetic other than integer multiplication and isqrt. Interval square roots follow endpointwise because √ is increasing.

Sine and cosine. These are obtained from mpmath's validated-interval context at 300 bits, which itself rounds outward. Its endpoints are binary floats and are converted to rationals exactly, by reading the (sign, mantissa, exponent) triple and forming man·2^exp — a decimal string would round silently, so it is never used. The result is then padded outward by a further 10^{-80}, so the enclosure remains valid even if the library's endpoint convention were to change.

Arcsine, by certify-by-inversion. The interval context provides no arcsine, and writing an approximate one would place an unverified number on the proof path. Instead the value is guessed and then proved, using only the verified sine and the monotonicity of sin on $[-\pi/2, \pi/2]$:

```
given rational y in [-1, 1]:
  t0  <- high-precision floating guess at asin(y)      # untrusted
  d   <- 10^-80
  repeat up to 60 times:
      lo, hi <- t0 - d, t0 + d
      if [lo, hi] subset of [-pi/2, pi/2]
         and sin_ivl(lo) <= y                          # verified inequality
         and sin_ivl(hi) >= y:                         # verified inequality
             return [lo, hi]                           # a proved bracket
      d <- 1000 d
  fail loudly
```

The two inequalities are checked with predicates that hold for every point of the enclosing intervals, so if the bracket is returned then $\sin(lo) \le y \le \sin(hi)$, and monotonicity gives lo ≤ arcsin(y) ≤ hi. The guess never has to be trusted, only to be close: a poor guess costs iterations and precision, never soundness, and if no bracket verifies the routine raises rather than returning. Termination is bounded by the loop count, and failure is loud.

What this means precisely. Binary floating point appears at exactly one place — the initial guess t0 above, and inside mpmath's outward-rounded interval routines whose endpoints are then converted exactly and padded. No floating-point value is ever accepted into an enclosure without a verifying inequality or an outward rounding. Floating point is used freely off the proof path, for the grid measurements of §3.3 and for Figure 1, and those results are labelled as measurements rather than as bounds.

One defect in the kernel as archived with version 1 of this note violated the enclosure invariant on the first conversions of a run. It was found by the consistency gate of §8, which reports it in full together with the fix; every value quoted in this note has been re-verified with the corrected kernel.

5. Hansen's slivers, and a settled discrepancy

Hansen's two removed regions have areas a_2 and a_3 in the tangent-chase recursion

$$
\begin{aligned}
x_0 &= 1 - s/2,\\
x_{i+1} &= 2x_i^2 / \left(1 - s\cdot x_i + \sqrt{1 - 2s\cdot x_i - x_i^2}\right),\\
a_i &= x_i\, x_{i+1}/4 - (\theta - \sin\theta)/2, \quad \theta = 2\cdot\arcsin(d/2),\\
d &= \sqrt{x_{i+1}^2/4 + (x_i + (s/2)\cdot x_{i+1})^2}.
\end{aligned}
$$

Evaluated as in §4, every digit of Table 1 of Baez–Bagdasaryan–Gibbs is reproduced, for both the x_i and the a_i; in particular $a_2 = 3.750723412843\cdot10^{-11}$ and $a_3 = 8.454119457933\cdot10^{-21}$.

The published table and the account of the same computation on the Azimuth blog disagree about a_3, the blog giving $8.4460\cdot10^{-21}$ — a difference in the third significant figure. The recursion is closed-form, so the disagreement is decidable. The enclosure obtained here contains the published value and excludes the blog's, which settles it in favour of the paper.

The mechanism behind Hansen's original error is visible in the same computation. Written in its unrationalised form,

$$x_{i+1} = \left[\, 1 - s\cdot x_i - \sqrt{1 - 2s\cdot x_i - x_i^2}\,\right] / 2,$$

the recursion subtracts two nearly equal quantities. In interval arithmetic this does not produce a silently wrong answer; it produces a visibly wide one. At x_4 the unrationalised form yields an enclosure of width $6.7\cdot10^{-60}$ against $1.0\cdot10^{-71}$ for the rationalised form — a widening by a factor of $6.55\cdot10^{11}$, growing monotonically along the recursion. Evaluating the unrationalised form by hand is a sufficient explanation for an answer three orders of magnitude too large, and is a small argument for interval arithmetic over careful floating point: the method reports its own loss of information.

6. The 2015 cover

The Baez–Bagdasaryan–Gibbs cover is built from the hexagon H and its copy H′ rotated by 30° + σ: the corner triangles of H \ H′ at C and E are removed (a slanted Pál step), the corner at A is reduced by two unit circles centred at cut points O and N (a slanted

Sprague step), and a further curvilinear triangle is removed. Two facts, both found the hard way, govern the reconstruction.

First, Sprague's tangency does not break at $\sigma > 0$ — it moves. The centres O and N lie on hexagon sides, and opposite sides are at distance exactly 1, so each unit circle is tangent to the opposite edge at the perpendicular foot, for every slant. This exact tangency is also why pixel-grid identification fails on this rung: near a tangency a boolean mask shatters into slivers. Second, the paper's prose does not match its program on one point: the label L is described as "another point where the rotated hexagon intersects the original hexagon", but in the published Java [12] it is the perpendicular foot of the cut point A_3 on the opposite side D_1C_1 — not a vertex of $H \cap H'$ at all. The program is the authority for the published number, and the program is what is certified here.

Mirrored statement for statement, the program's own gates all pass: at $\sigma = 1.3$ the area reproduces all sixteen printed digits (0.8441153768593765) together with both published side-condition angles (90.00593°, 122.9277°); at Egan's $\sigma = 1.294389444703601012°$ the area agrees with his eighteen digits to one unit in the last place; and as $\sigma \to 0$ the cover satisfies cover = Sprague − lens exactly, tying this rung to §3. Two observations fall out. Egan's eighteen-digit σ is precisely the root of the first side condition reaching a right angle — the optimum sits where the constraint becomes active, which explains his value rather than merely matching it. And the certified evaluation shows the paper's printed figure to be its double-precision rounding of 0.84411537685937674680…, agreement exactly at the limit of Java's arithmetic.

In the kernel, with both slants as exact rationals and the side conditions certified as dot-product signs, the construction yields two unconditional certified bounds:

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a ≤ 0.844115376859376746806104420762…   (σ = 13/10)
a ≤ 0.844115297128419059214192192376…   (Egan's σ)
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the second with its active condition certified at margin 4.4×10^{-22} — Egan rounded his last digit the right way.

7. The 2018 record

Gibbs's record cover extends the 2015 construction by three further removals, and its boundary near the corner A includes a curve that the paper describes only constructively — the reason, I believe, the record went unverified for eight years. The published program [11] resolves it: the curve is the image of a straight chord under a five-step ruler-and-compass map — three radial points, one triangulation, one reflection in the line through two edge midpoints — sampled at 20000 points. The map contains square roots and nothing else.

The mirror reproduces the record's own optimisation: minimum area 0.844093594386 at slant 1.54940°, whose ten-digit ceiling is the published 0.8440935944. Refining the sampling shows clean $1/k^2$ convergence from below (0.844093594383704, …386462, …386488 at $k = 2\cdot10^3, 2\cdot10^4, 2\cdot10^5$), so the construction's true area is ≈ 0.84409359438649 and the published ceiling holds it with $\approx 1.35\times10^{-11}$ to spare. The discretised construction is certified in the kernel at $\sigma = 7747/5000$, $k = 2000$: 0.8440935943837045008975473749105925, enclosure width 7×10^{-52}.

A record certificate must bound the true area, and the gap between it and any discretisation is the lens between each curve piece and its chord: writing γ for the curve, $|\text{lens}| \le c \cdot (1/8)(\Delta s)^2 \cdot \sup\|\gamma''\|$ on each piece, with no assumption about which side of its chord the curve bulges. The derivative bounds come from running the five-step map in second-order jet arithmetic over interval coefficients — triples (value, d/ds, d^2/ds^2) composed through $+, -, \times, \div, \sqrt{}$: exact forward-mode differentiation with outward rounding. A profile over 256 parameter subintervals certifies $\sup\|\gamma''\| \le 0.3996$ (a 64-subinterval version gives 1.75, pure interval dependency — and a first attempt using that global bound fails by 1.8×10^{-11}, which is worth recording), and the localised form $\Sigma\, M_1M_2/256 = 0.053778$ gives, at $k = 39937$ certified samples,

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true area ≤ polygon + six exact unit arcs + 4.215×10⁻¹²,
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whose upper endpoint lies below the published ceiling with certified margin 9.3×10^{-12}. Every boundary arc other than the curve is exact by the same opposite-sides tangency as everywhere in this ladder. To my knowledge this is the first verification of the record since its publication:

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a ≤ 0.8440935944.
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8. A soundness incident

Version 1 of this note stated that no floating-point value enters an enclosure without verification. The kernel as then archived had a defect in that claim, and it is worth reporting in full because of how it was caught.

The kernel's mpf-to-rational conversion re-created each value in the ambient mpmath context before reading its mantissa. A fresh process holds 53 bits, so the first conversions of every run were silently rounded to double precision — a one-sided error of order 10^{-17}, far outside the 10^{-80} outward pad — while every later conversion was exact, because the first arcsine call raises the working precision. A first-call-poisoned, self-healing kernel: almost every run looks perfect.

It was caught by a consistency gate, not by inspection. The 2015 cover at $\sigma = 0$ must satisfy cover + lens = Sprague exactly (segment additivity on a common circle); the kernel missed zero by 8.1×10^{-19} at interval width 10^{-58}. The rungs certified earlier call arcsine before π — the safe ordering, by luck — and all their published digits were re-confirmed both by independent 50-digit arbitration and by re-runs after the fix, which reads the raw mantissa tuple and converts nothing. The code archived with this version carries the corrected kernel.

The lesson is structural. A single certified computation cannot detect a primitive that errs self-consistently; two independently certified rungs joined by an exact identity can. Ladders should be built with their gates.

9. Results

rung	area	enclosure width
hexagon	$\sqrt{3}/2$	exact
Pál 1920	$2 - 2/\sqrt{3}$	$6.7\cdot10^{-61}$
Sprague 1936	0.844137708435197570894066994044…	$7.9\cdot10^{-60}$

Hansen 1992	0.844137708397690336757182874586…	$7.9\cdot10^{-60}$
BBG 2015, σ = 1.3	0.844115376859376746806104420762…	below 10^{-33}
BBG 2015, Egan's σ	0.844115297128419059214192192376…	below 10^{-33}
Gibbs 2018 (record)	a ≤ 0.8440935944	slack $4.2\cdot10^{-12}$, margin $9.3\cdot10^{-12}$

The five incidence conditions underlying the reconstruction,

$$|T - E_3| = |P - E_3| = |X - E_3| = |X - C_2| = |Q - C_2| = 1,$$

each hold with an enclosure of width about 10^{-60} about the value 1, so the geometry is verified rather than assumed.

Reconciliation with the quoted figures. The literature quotes 0.844137708436 for Sprague's cover and 0.844137708398 for Hansen's. (Neither appears in Sprague, per §3.1; Hansen's does appear in his own paper.) Both are upper bounds, and both are the values above rounded up at twelve decimals: the twelve-decimal ceiling of each enclosure reproduces the quoted figure exactly. There is no disagreement — the quoted numbers are conservative roundings of the quantities computed here. The same holds up the ladder: the 2015 paper's 0.8441153768593765 is the Java double's rounding of the certified …67468…, Egan's eighteen digits are exact, and Gibbs's ceiling 0.8440935944 is confirmed as a true upper bound of his construction.

10. Scope

Only areas are certified. Universality — that each set covers every set of diameter 1 — is taken from the cited literature for every rung. For the 2015 cover the two computational side conditions on which its universality argument relies are certified here at the slants used; the corresponding case checks for the 2018 record are Gibbs's. The record bound is certified at the single slant σ = 7747/5000, which suffices for an upper bound. No optimality is claimed for any construction, and no new upper bound is claimed: the contribution is that the existing ladder, from 1914 to the record, now carries certificates.

11. Reproducibility

lebesgue/ladder_base.py — hexagon and Pál. sprague_identify.py, sprague_component.py, sprague_exact.py, sprague_certified.py — §3, as in version 1; lebesgue/figure_sprague.py emits Figure 1. verify-upper/hansen_table.py — §5. lebesgue/bbg_analytic.py — the 2015 mirror and its gates; bbg_certified.py — the certified 2015 bounds and the σ = 0 consistency gate of §8. lebesgue/gibbs_mirror.py — the record mirror and its sweep; gibbs_certified.py — the certified discretised evaluation; gibbs_record.py — the jet arithmetic and the record certificate. core/exact.py — the arithmetic kernel, carrying the fix of §8. Each certifying script prints its enclosures and the comparison it is meant to reproduce, and exits non-zero if a check fails.

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